

Lowering-Raising triples and $U_q(\mathfrak{sl}_2)$

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Abstract

We introduce the notion of a lowering-raising (or LR) triple of linear transformations on a nonzero finite-dimensional vector space. We show how to normalize an LR triple, and classify up to isomorphism the normalized LR triples. We describe the LR triples using various maps, such as the reflectors, the inverters, the unipotent maps, and the rotators. We relate the LR triples to the equitable presentation of the quantum algebra $U_q(\mathfrak{sl}_2)$ and Lie algebra \mathfrak{sl}_2 .

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1 Introduction

For the quantum algebra $U_q(\mathfrak{sl}_2)$, the equitable presentation was introduced in [18] and further investigated in [23], [24]. For the Lie algebra \mathfrak{sl}_2 , the equitable presentation was introduced in [12] and comprehensively studied in [5]. These equitable presentations have been related to Leonard pairs [1], [2], tridiagonal pairs [6], Leonard triples [9], [13], the universal Askey-Wilson algebra [22], the tetrahedron algebra [11], [12], [16], the q -tetrahedron algebra [14], [17], and distance-regular graphs [25]. See also [3], [8], [15], [21].

From the equitable point of view, consider a finite-dimensional irreducible module for $U_q(\mathfrak{sl}_2)$ or \mathfrak{sl}_2 . In [23, Lemma 7.3] and [5, Section 8] we encounter three nilpotent linear transformations of the module, with each transformation acting as a lowering map and raising map in multiple ways. In order to describe this situation more precisely, we now introduce the notion of a lowering-raising (or LR) triple of linear transformations.

An LR triple is described as follows (formal definitions begin in Section 2). Fix an integer $d \geq 0$. Let \mathbb{F} denote a field, and let V denote a vector space over \mathbb{F} with dimension $d+1$. By a decomposition of V we mean a sequence $\{V_i\}_{i=0}^d$ of one-dimensional subspaces whose direct sum is V . Let $\{V_i\}_{i=0}^d$ denote a decomposition of V . A linear transformation $A \in \text{End}(V)$ is said to lower $\{V_i\}_{i=0}^d$ whenever $AV_i = V_{i-1}$ for $1 \leq i \leq d$ and $AV_0 = 0$. The map A is said to raise $\{V_i\}_{i=0}^d$ whenever $AV_i = V_{i+1}$ for $0 \leq i \leq d-1$ and $AV_d = 0$. An ordered pair of elements A, B in $\text{End}(V)$ is called lowering-raising (or LR) whenever there exists a decomposition of V that is lowered by A and raised by B . A 3-tuple of elements A, B, C in

$\text{End}(V)$ is called an LR triple whenever any two of A, B, C form an LR pair on V . The LR triple A, B, C is said to be over \mathbb{F} and have diameter d .

In this paper we obtain three main results, which are summarized as follows: (i) we show how to normalize an LR triple, and classify up to isomorphism the normalized LR triples; (ii) we describe the LR triples using various maps, such as the reflectors, the inverters, the unipotent maps, and the rotators; (iii) we relate the LR triples to the equitable presentations of $U_q(\mathfrak{sl}_2)$ and \mathfrak{sl}_2 .

We now describe our results in more detail. We set the stage with some general remarks; the assertions therein will be established in the main body of the paper. Let the integer d and the vector space V be as above, and assume for the moment that $d = 0$. Then $A, B, C \in \text{End}(V)$ form an LR triple if and only if each of A, B, C is zero; this LR triple is called trivial. Until further notice, assume that $d \geq 0$ and let A, B, C denote an LR triple on V . As we describe this LR triple, we will use the following notation. Observe that any permutation of A, B, C is an LR triple on V . For any object f that we associate with A, B, C let f' (resp. f'') denote the corresponding object for the LR triple B, C, A (resp. C, A, B). Since A, B is an LR pair on V , there exists a decomposition $\{V_i\}_{i=0}^d$ of V that is lowered by A and raised by B . This decomposition is uniquely determined by A, B and called the (A, B) -decomposition of V . For $0 \leq i \leq d$ we have $A^{d-i}V = V_0 + V_1 + \cdots + V_i$ and $B^{d-i}V = V_d + V_{d-1} + \cdots + V_{d-i}$.

We now introduce the parameter array of A, B, C . For $1 \leq i \leq d$ we have $AV_i = V_{i-1}$ and $BV_{i-1} = V_i$. Therefore, V_i is invariant under BA and the corresponding eigenvalue is a nonzero scalar in \mathbb{F} . Denote this eigenvalue by φ_i . For notational convenience define $\varphi_0 = 0$ and $\varphi_{d+1} = 0$. We call the sequence

$$(\{\varphi_i\}_{i=1}^d; \{\varphi'_i\}_{i=1}^d; \{\varphi''_i\}_{i=1}^d)$$

the parameter array of A, B, C .

We now introduce the idempotent data of A, B, C . For $0 \leq i \leq d$ define $E_i \in \text{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ for $0 \leq j \leq d$, $j \neq i$. Thus E_i is the projection from V onto V_i . Note that $V_i = E_i V$. We have

$$E_i = \frac{A^{d-i} B^d A^i}{\varphi_1 \cdots \varphi_d}, \quad E_i = \frac{B^i A^d B^{d-i}}{\varphi_1 \cdots \varphi_d}.$$

We call the sequence

$$(\{E_i\}_{i=0}^d; \{E'_i\}_{i=0}^d; \{E''_i\}_{i=0}^d)$$

the idempotent data of A, B, C .

We now introduce the Toeplitz data of A, B, C . A basis $\{v_i\}_{i=0}^d$ of V is called an (A, B) -basis whenever $v_i \in V_i$ for $0 \leq i \leq d$ and $Av_i = v_{i-1}$ for $1 \leq i \leq d$. Let $\{u_i\}_{i=0}^d$ denote a (C, B) -basis of V and let $\{v_i\}_{i=0}^d$ denote a (C, A) -basis of V such that $u_0 = v_0$. Let T denote

the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$. Then T has the form

$$T = \begin{pmatrix} \alpha_0 & \alpha_1 & \cdot & \cdot & \cdot & \alpha_d \\ & \alpha_0 & \alpha_1 & \cdot & \cdot & \cdot \\ & & \alpha_0 & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \alpha_1 \\ \mathbf{0} & & & & & \alpha_0 \end{pmatrix},$$

where $\alpha_i \in \mathbb{F}$ for $0 \leq i \leq d$ and $\alpha_0 = 1$. A matrix of the above form is said to be upper triangular and Toeplitz, with parameters $\{\alpha_i\}_{i=0}^d$. The matrix T^{-1} is upper triangular and Toeplitz; let $\{\beta_i\}_{i=0}^d$ denote its parameters. We call the sequence

$$(\{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^d; \{\alpha'_i\}_{i=0}^d, \{\beta'_i\}_{i=0}^d; \{\alpha''_i\}_{i=0}^d, \{\beta''_i\}_{i=0}^d)$$

the Toeplitz data of A, B, C .

We now introduce the trace data of A, B, C . For $0 \leq i \leq d$ let a_i denote the trace of CE_i . We have $\sum_{i=0}^d a_i = 0$. If A, B, C is trivial then $a_0 = 0$. If A, B, C is nontrivial then $a_i = \alpha'_1(\varphi''_{d-i+1} - \varphi''_{d-i})$ and $a_i = \alpha'_1(\varphi'_{d-i+1} - \varphi'_{d-i})$ for $0 \leq i \leq d$. We call the sequence

$$(\{a_i\}_{i=0}^d; \{a'_i\}_{i=0}^d; \{a''_i\}_{i=0}^d)$$

the trace data of A, B, C .

With respect to an (A, B) -basis of V , the matrices representing A, B, C are

$$A : \begin{pmatrix} 0 & 1 & & & \mathbf{0} \\ & 0 & 1 & & \\ & & 0 & \cdot & \\ & & & \cdot & \cdot \\ & & & & \cdot & 1 \\ \mathbf{0} & & & & & 0 \end{pmatrix}, \quad B : \begin{pmatrix} 0 & & & & \mathbf{0} \\ \varphi_1 & 0 & & & \\ & \varphi_2 & 0 & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ \mathbf{0} & & & & \varphi_d & 0 \end{pmatrix},$$

$$C : \begin{pmatrix} a_0 & \varphi'_d/\varphi_1 & & & \mathbf{0} \\ \varphi''_d & a_1 & \varphi'_{d-1}/\varphi_2 & & \\ & \varphi''_{d-1} & a_2 & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \varphi'_1/\varphi_d \\ \mathbf{0} & & & & \varphi''_1 & a_d \end{pmatrix}.$$

Assume for the moment that A, B, C is nontrivial. Then A, B, C is determined up to isomorphism by its parameter array and any one of

$$a_0, a'_0, a''_0; \quad a_d, a'_d, a''_d; \quad \alpha_1, \alpha'_1, \alpha''_1; \quad \beta_1, \beta'_1, \beta''_1.$$

We often put the emphasis on α_1 , and call this the first Toeplitz number of A, B, C . In Propositions 15.12–15.14 we obtain some recursions that give the Toeplitz data of A, B, C in terms of its parameter array and first Toeplitz number.

We now introduce the bipartite condition. The LR triple A, B, C is said to be bipartite whenever $a_i = a'_i = a''_i = 0$ for $0 \leq i \leq d$. Assume for the moment that A, B, C is not bipartite. Then A, B, C is nontrivial, and each of

$$\alpha_1, \quad \alpha'_1, \quad \alpha''_1, \quad \beta_1, \quad \beta'_1, \quad \beta''_1$$

is nonzero. Until further notice assume that A, B, C is bipartite. Then $d = 2m$ is even. Moreover for $0 \leq i \leq d$, each of

$$\alpha_i, \quad \alpha'_i, \quad \alpha''_i, \quad \beta_i, \quad \beta'_i, \quad \beta''_i$$

is zero if i is odd and nonzero if i is even. There exists a direct sum $V = V_{\text{out}} + V_{\text{in}}$ such that V_{out} is equal to each of

$$\sum_{j=0}^m E_{2j} V, \quad \sum_{j=0}^m E'_{2j} V, \quad \sum_{j=0}^m E''_{2j} V$$

and V_{in} is equal to each of

$$\sum_{j=0}^{m-1} E_{2j+1} V, \quad \sum_{j=0}^{m-1} E'_{2j+1} V, \quad \sum_{j=0}^{m-1} E''_{2j+1} V.$$

The dimensions of V_{out} and V_{in} are $m+1$ and m , respectively. We have

$$\begin{aligned} AV_{\text{out}} &= V_{\text{in}}, & BV_{\text{out}} &= V_{\text{in}}, & CV_{\text{out}} &= V_{\text{in}}, \\ AV_{\text{in}} &\subseteq V_{\text{out}}, & BV_{\text{in}} &\subseteq V_{\text{out}}, & CV_{\text{in}} &\subseteq V_{\text{out}}. \end{aligned}$$

Define

$$A_{\text{out}}, \quad A_{\text{in}}, \quad B_{\text{out}}, \quad B_{\text{in}}, \quad C_{\text{out}}, \quad C_{\text{in}} \tag{1}$$

in $\text{End}(V)$ as follows. The map A_{out} acts on V_{out} as A , and on V_{in} as zero. The map A_{in} acts on V_{in} as A , and on V_{out} as zero. The other maps in (1) are similarly defined. By construction

$$A = A_{\text{out}} + A_{\text{in}}, \quad B = B_{\text{out}} + B_{\text{in}}, \quad C = C_{\text{out}} + C_{\text{in}}.$$

We are done assuming that A, B, C is bipartite.

We now introduce the equitable condition. The LR triple A, B, C is said to be equitable whenever $\alpha_i = \alpha'_i = \alpha''_i$ for $0 \leq i \leq d$. In this case $\beta_i = \beta'_i = \beta''_i$ for $0 \leq i \leq d$. Assume for the moment that A, B, C is trivial. Then A, B, C is equitable. Next assume that A, B, C is nonbipartite. Then A, B, C is equitable if and only if $\alpha_1 = \alpha'_1 = \alpha''_1$. In this case $\varphi_i = \varphi'_i = \varphi''_i$ for $1 \leq i \leq d$, and $a_i = a'_i = a''_i = \alpha_1(\varphi_{d-i+1} - \varphi_{d-i})$ for $0 \leq i \leq d$. Next assume that A, B, C is bipartite and nontrivial. Then A, B, C is equitable if and only if $\alpha_2 = \alpha'_2 = \alpha''_2$. In this case $\varphi_{i-1}\varphi_i = \varphi'_{i-1}\varphi'_i = \varphi''_{i-1}\varphi''_i$ for $2 \leq i \leq d$. We are done with our general remarks.

Concerning the normalization of LR triples, we now define what it means for A, B, C to be normalized. Assume for the moment that A, B, C is trivial. Then A, B, C is normalized.

Next assume that A, B, C is nonbipartite. Then A, B, C is normalized whenever $\alpha_1 = \alpha'_1 = \alpha''_1 = 1$. Next assume that A, B, C is bipartite and nontrivial. Then A, B, C is normalized whenever $\alpha_2 = \alpha'_2 = \alpha''_2 = 1$. If A, B, C is normalized then A, B, C is equitable. We now explain how to normalize A, B, C . Assume for the moment that A, B, C is trivial. Then there is nothing to do. Next assume that A, B, C is nonbipartite. Then there exists a unique sequence α, β, γ of nonzero scalars in \mathbb{F} such that $\alpha A, \beta B, \gamma C$ is normalized. Next assume that A, B, C is bipartite and nontrivial. Then there exists a unique sequence α, β, γ of nonzero scalars in \mathbb{F} such that

$$\alpha A_{\text{out}} + A_{\text{in}}, \quad \beta B_{\text{out}} + B_{\text{in}}, \quad \gamma C_{\text{out}} + C_{\text{in}}$$

is normalized.

We now describe our classification up to isomorphism of the normalized LR triples over \mathbb{F} . Up to isomorphism there exists a unique normalized LR triple over \mathbb{F} with diameter $d = 0$, and this LR triple is trivial. Up to isomorphism there exists a unique normalized LR triple over \mathbb{F} with diameter $d = 1$, and this is given in Lemma 24.2. For $d \geq 2$, we display nine families of normalized LR triples over \mathbb{F} that have diameter d , denoted

$$\begin{aligned} & \text{NBWeyl}_d^+(\mathbb{F}; j, q), & \text{NBWeyl}_d^-(\mathbb{F}; j, q), & \text{NBWeyl}_d^-(\mathbb{F}; t), \\ & \text{NBG}_d(\mathbb{F}; q), & \text{NBG}_d(\mathbb{F}; 1), & \\ & \text{NBNG}_d(\mathbb{F}; t), & & \\ & \text{B}_d(\mathbb{F}; t, \rho_0, \rho'_0, \rho''_0), & \text{B}_d(\mathbb{F}; 1, \rho_0, \rho'_0, \rho''_0), & \text{B}_2(\mathbb{F}; \rho_0, \rho'_0, \rho''_0). \end{aligned}$$

We show that each normalized LR triple over \mathbb{F} with diameter d is isomorphic to exactly one of these examples.

We now describe the LR triples using various maps. Let A, B, C denote an LR triple on V . We show that there exists a unique antiautomorphism \dagger of $\text{End}(V)$ that sends $A \leftrightarrow B$. We call \dagger the (A, B) -reflector. Assume for the moment that A, B, C is equitable and nonbipartite. We show that \dagger fixes C . Next assume that A, B, C is equitable, bipartite, and nontrivial. We show that \dagger sends $A_{\text{out}} \leftrightarrow B_{\text{in}}$ and $B_{\text{out}} \leftrightarrow A_{\text{in}}$. We also show that \dagger sends each of $C_{\text{out}}, C_{\text{in}}$ to a scalar multiple of the other. Define

$$\Psi = \sum_{i=0}^d \frac{\varphi_1 \varphi_2 \cdots \varphi_i}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-i+1}} E_i.$$

We call Ψ the (A, B) -inverter. We show that the following three LR pairs are mutually isomorphic:

$$A, \Psi^{-1} B \Psi \quad B, A \quad \Psi A \Psi^{-1}, B.$$

Define

$$\mathbb{A} = \sum_{i=0}^d E_{d-i} E''_i, \quad \mathbb{B} = \sum_{i=0}^d E'_{d-i} E_i, \quad \mathbb{C} = \sum_{i=0}^d E''_{d-i} E'_i.$$

We call $\mathbb{A}, \mathbb{B}, \mathbb{C}$ the unipotent maps for A, B, C . We show that

$$\mathbb{A} = \sum_{i=0}^d \alpha'_i A^i, \quad \mathbb{B} = \sum_{i=0}^d \alpha''_i B^i, \quad \mathbb{C} = \sum_{i=0}^d \alpha_i C^i$$

and

$$\mathbb{A}^{-1} = \sum_{i=0}^d \beta'_i A^i, \quad \mathbb{B}^{-1} = \sum_{i=0}^d \beta''_i B^i, \quad \mathbb{C}^{-1} = \sum_{i=0}^d \beta_i C^i.$$

By a rotator for A, B, C we mean an element $R \in \text{End}(V)$ such that for $0 \leq i \leq d$,

$$E_i R = R E'_i, \quad E_i R = R E''_i, \quad E''_i R = R E_i.$$

Let \mathcal{R} denote the set of rotators for A, B, C . Note that \mathcal{R} is a subspace of the \mathbb{F} -vector space $\text{End}(V)$. We obtain the following basis for \mathcal{R} . Assume for the moment that A, B, C is trivial. Then $\mathcal{R} = \text{End}(V)$ has a basis consisting of the identity element. Next assume that A, B, C is nonbipartite. Then \mathcal{R} has a basis Ω such that

$$\begin{aligned} \Omega &= \mathbb{B} \left(\sum_{i=0}^d \frac{\varphi_1 \cdots \varphi_i}{\varphi_d \cdots \varphi_{d-i+1}} E_i \right) \mathbb{A} = \mathbb{C} \left(\sum_{i=0}^d \frac{\varphi_1 \cdots \varphi_i}{\varphi_d \cdots \varphi_{d-i+1}} E'_i \right) \mathbb{B} \\ &= \mathbb{A} \left(\sum_{i=0}^d \frac{\varphi_1 \cdots \varphi_i}{\varphi_d \cdots \varphi_{d-i+1}} E''_i \right) \mathbb{C}. \end{aligned}$$

Next assume that A, B, C is bipartite and nontrivial. Then \mathcal{R} has a basis $\Omega_{\text{out}}, \Omega_{\text{in}}$ such that

$$\begin{aligned} \Omega_{\text{out}} &= \mathbb{B} \left(\sum_{j=0}^{d/2} \frac{\varphi_1 \varphi_2 \cdots \varphi_{2j}}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-2j+1}} E_{2j} \right) \mathbb{A} = \mathbb{C} \left(\sum_{j=0}^{d/2} \frac{\varphi_1 \varphi_2 \cdots \varphi_{2j}}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-2j+1}} E'_{2j} \right) \mathbb{B} \\ &= \mathbb{A} \left(\sum_{j=0}^{d/2} \frac{\varphi_1 \varphi_2 \cdots \varphi_{2j}}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-2j+1}} E''_{2j} \right) \mathbb{C} \end{aligned}$$

and

$$\begin{aligned} \Omega_{\text{in}} &= \mathbb{B} \left(\sum_{j=0}^{d/2-1} \frac{\varphi_2 \varphi_3 \cdots \varphi_{2j+1}}{\varphi_{d-1} \varphi_{d-2} \cdots \varphi_{d-2j}} E_{2j+1} \right) \mathbb{A} = \mathbb{C} \left(\sum_{j=0}^{d/2-1} \frac{\varphi_2 \varphi_3 \cdots \varphi_{2j+1}}{\varphi_{d-1} \varphi_{d-2} \cdots \varphi_{d-2j}} E'_{2j+1} \right) \mathbb{B} \\ &= \mathbb{A} \left(\sum_{j=0}^{d/2-1} \frac{\varphi_2 \varphi_3 \cdots \varphi_{2j+1}}{\varphi_{d-1} \varphi_{d-2} \cdots \varphi_{d-2j}} E''_{2j+1} \right) \mathbb{C}. \end{aligned}$$

We now briefly relate the LR triples to the equitable presentations of $U_q(\mathfrak{sl}_2)$ and \mathfrak{sl}_2 . Adjusting the equitable presentation of $U_q(\mathfrak{sl}_2)$ in two ways, we obtain an algebra $U_q^R(\mathfrak{sl}_2)$ called the reduced $U_q(\mathfrak{sl}_2)$ algebra, and an algebra $U_q^E(\mathfrak{sl}_2)$ called the extended $U_q(\mathfrak{sl}_2)$ algebra. Let A, B, C denote an LR triple on V . After imposing some minor restrictions on its parameter

array, we use A, B, C to construct a module on V for $U_q(\mathfrak{sl}_2)$ or $U_q^R(\mathfrak{sl}_2)$ or $U_q^E(\mathfrak{sl}_2)$ or \mathfrak{sl}_2 . Each construction involves the equitable presentation.

This paper is organized as follows. In Section 2 we review some basic concepts and explain our notation. In Sections 3–10 we develop a theory of LR pairs that will be applied to LR triples later in the paper. In Section 11 we classify a type of finite sequence said to be constrained, for use in our LR triple classification later in the paper. Section 12 is about upper triangular Toeplitz matrices. In Section 13 we introduce the LR triples, and discuss their parameter array, idempotent data, Toeplitz data, and trace data. In Sections 14, 15 we obtain some equations relating the parameter array, Toeplitz data, and trace data. We also introduce the LR triples of Weyl and q -Weyl type. Sections 16–18 are about the bipartite, equitable, and normalized LR triples, respectively. In Sections 19, 20 we compare the structure of a bipartite and nonbipartite LR triple, using the notions of an idempotent centralizer and double lowering space. Sections 21–23 are about the unipotent maps, rotators, and reflectors, respectively. In Sections 24–30 we classify up to isomorphism the normalized LR triples. Section 31 is about the Toeplitz data, and how the unipotent maps are related to the exponential function and quantum exponential function. In Section 32 we display some relations that are satisfied by an LR triple. In Section 33 we relate the LR triples to the equitable presentations of $U_q(\mathfrak{sl}_2)$ and \mathfrak{sl}_2 . Section 34 contains three characterizations of an LR triple. Sections 35, 36 are appendices that contain some matrix representations of an LR triple.

2 Preliminaries

We now begin our formal argument. In this section we review some basic concepts and explain our notation. We will be discussing algebras and Lie algebras. An algebra without the Lie prefix is meant to be associative and have a 1. A subalgebra has the same 1 as the parent algebra. Recall the ring of integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. Throughout the paper we fix an integer $d \geq 0$. For a sequence $\{u_i\}_{i=0}^d$, we call u_i the i -component or i -coordinate of the sequence. By the *inversion* of $\{u_i\}_{i=0}^d$ we mean the sequence $\{u_{d-i}\}_{i=0}^d$. Let \mathbb{F} denote a field. Let V denote a vector space over \mathbb{F} with dimension $d+1$. Let $\text{End}(V)$ denote the \mathbb{F} -algebra consisting of the \mathbb{F} -linear maps from V to V . Let $\text{Mat}_{d+1}(\mathbb{F})$ denote the \mathbb{F} -algebra consisting of the $d+1$ by $d+1$ matrices that have all entries in \mathbb{F} . We index the rows and columns by $0, 1, \dots, d$. Let $\{v_i\}_{i=0}^d$ denote a basis for V . For $A \in \text{End}(V)$ and $M \in \text{Mat}_{d+1}(\mathbb{F})$, we say that M *represents* A with respect to $\{v_i\}_{i=0}^d$ whenever $Av_j = \sum_{i=0}^d M_{ij}v_i$ for $0 \leq j \leq d$. Suppose we are given two bases for V , denoted $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$. By the *transition matrix from* $\{u_i\}_{i=0}^d$ *to* $\{v_i\}_{i=0}^d$ we mean the matrix $S \in \text{Mat}_{d+1}(\mathbb{F})$ such that $v_j = \sum_{i=0}^d S_{ij}u_i$ for $0 \leq j \leq d$. Let S denote the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$. Then S^{-1} exists and equals the transition matrix from $\{v_i\}_{i=0}^d$ to $\{u_i\}_{i=0}^d$. Let $\{w_i\}_{i=0}^d$ denote a basis for V and let H denote the transition matrix from $\{v_i\}_{i=0}^d$ to $\{w_i\}_{i=0}^d$. Then SH is the transition matrix from $\{u_i\}_{i=0}^d$ to $\{w_i\}_{i=0}^d$. Let $A \in \text{End}(V)$ and let $M \in \text{Mat}_{d+1}(\mathbb{F})$ represent A with respect to $\{u_i\}_{i=0}^d$. Then $S^{-1}MS$ represents A with respect to $\{v_i\}_{i=0}^d$. Define a matrix

$\mathbf{Z} \in \text{Mat}_{d+1}(\mathbb{F})$ with (i, j) -entry $\delta_{i+j, d}$ for $0 \leq i, j \leq d$. For example if $d = 3$,

$$\mathbf{Z} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $\mathbf{Z}^2 = I$. Let $\{v_i\}_{i=0}^d$ denote a basis for V and consider the inverted basis $\{v_{d-i}\}_{i=0}^d$. Then \mathbf{Z} is the transition matrix from $\{v_i\}_{i=0}^d$ to $\{v_{d-i}\}_{i=0}^d$.

By a *decomposition* of V we mean a sequence $\{V_i\}_{i=0}^d$ of one dimensional subspaces of V such that $V = \sum_{i=0}^d V_i$ (direct sum). Given a decomposition $\{V_i\}_{i=0}^d$ of V , for notational convenience define $V_{-1} = 0$ and $V_{d+1} = 0$. Let $\{v_i\}_{i=0}^d$ denote a basis for V . For $0 \leq i \leq d$ let V_i denote the span of v_i . Then the sequence $\{V_i\}_{i=0}^d$ is a decomposition of V , said to be *induced* by the basis $\{v_i\}_{i=0}^d$. Let $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$ denote bases for V . Then the following are equivalent: (i) the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$ is diagonal; (ii) $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$ induce the same decomposition of V .

Let $\{V_i\}_{i=0}^d$ denote a decomposition of V . For $0 \leq i \leq d$ define $E_i \in \text{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ for $0 \leq j \leq d$, $j \neq i$. We call E_i the *i*th *primitive idempotent* for $\{V_i\}_{i=0}^d$. We have (i) $E_i E_j = \delta_{i,j} E_i$ ($0 \leq i, j \leq d$); (ii) $I = \sum_{i=0}^d E_i$; (iii) $V_i = E_i V$ ($0 \leq i \leq d$); (iv) $\text{rank}(E_i) = 1 = \text{tr}(E_i)$ ($0 \leq i \leq d$), where tr means trace. We call $\{E_i\}_{i=0}^d$ the *idempotent sequence* for $\{V_i\}_{i=0}^d$. Note that $\{E_{d-i}\}_{i=0}^d$ is the idempotent sequence for the decomposition $\{V_{d-i}\}_{i=0}^d$.

Let $\{v_i\}_{i=0}^d$ denote a basis for V . Let $\{V_i\}_{i=0}^d$ denote the induced decomposition of V , with idempotent sequence $\{E_i\}_{i=0}^d$. For $0 \leq r \leq d$ consider the matrix in $\text{Mat}_{d+1}(\mathbb{F})$ that represents E_r with respect to $\{v_i\}_{i=0}^d$. This matrix has (r, r) -entry 1 and all other entries 0.

Lemma 2.1. *Let $A \in \text{End}(V)$. Let $\{V_i\}_{i=0}^d$ denote a decomposition of V with idempotent sequence $\{E_i\}_{i=0}^d$. Consider a basis for V that induces $\{V_i\}_{i=0}^d$. Let $M \in \text{Mat}_{d+1}(\mathbb{F})$ represent A with respect to this basis. Then for $0 \leq r, s \leq d$ the entry $M_{r,s} = 0$ if and only if $E_r A E_s = 0$.*

Proof. Represent A, E_r, E_s by matrices with respect to the given basis. □

By a *flag* on V we mean a sequence $\{U_i\}_{i=0}^d$ of subspaces of V such that U_i has dimension $i + 1$ for $0 \leq i \leq d$ and $U_{i-1} \subseteq U_i$ for $1 \leq i \leq d$. For a flag $\{U_i\}_{i=0}^d$ on V we have $U_d = V$. Let $\{V_i\}_{i=0}^d$ denote a decomposition of V . For $0 \leq i \leq d$ define $U_i = V_0 + \cdots + V_i$. Then the sequence $\{U_i\}_{i=0}^d$ is a flag on V . This flag is said to be *induced* by the decomposition $\{V_i\}_{i=0}^d$. Let $\{u_i\}_{i=0}^d$ denote a basis of V . This basis induces a decomposition of V , which in turn induces a flag on V . This flag is said to be *induced* by the basis $\{u_i\}_{i=0}^d$. Let $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$ denote bases of V . Then the following are equivalent: (i) the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$ is upper triangular; (ii) $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$ induce the same flag on V .

Suppose we are given two flags on V , denoted $\{U_i\}_{i=0}^d$ and $\{U'_i\}_{i=0}^d$. These flags are called *opposite* whenever $U_i \cap U'_j = 0$ if $i + j < d$ ($0 \leq i, j \leq d$). The following are equivalent: (i)

$\{U_i\}_{i=0}^d$ and $\{U'_i\}_{i=0}^d$ are opposite; (ii) there exists a decomposition $\{V_i\}_{i=0}^d$ of V that induces $\{U_i\}_{i=0}^d$ and whose inversion induces $\{U'_i\}_{i=0}^d$. In this case $V_i = U_i \cap U'_{d-i}$ for $0 \leq i \leq d$.

Let $\{V_i\}_{i=0}^d$ denote a decomposition of V . For $A \in \text{End}(V)$, we say that A *lowers* $\{V_i\}_{i=0}^d$ whenever $AV_i = V_{i-1}$ for $1 \leq i \leq d$ and $AV_0 = 0$.

Lemma 2.2. *Let $\{V_i\}_{i=0}^d$ denote a decomposition of V , with idempotent sequence $\{E_i\}_{i=0}^d$. For $A \in \text{End}(V)$ the following are equivalent:*

- (i) A lowers $\{V_i\}_{i=0}^d$;
- (ii) $E_i A E_j = \begin{cases} \neq 0 & \text{if } j - i = 1; \\ 0 & \text{if } j - i \neq 1 \end{cases} \quad (0 \leq i, j \leq d).$

Proof. Use Lemma 2.1. □

Let $\{V_i\}_{i=0}^d$ denote a decomposition of V and let $A \in \text{End}(V)$. Assume that $\{V_i\}_{i=0}^d$ is lowered by A . Then $V_i = A^{d-i}V_d$ for $0 \leq i \leq d$. Moreover $A^{d+1} = 0$. For $0 \leq i \leq d$ the subspace $V_0 + \dots + V_i$ is the kernel of A^{i+1} and equal to $A^{d-i}V$. In particular, V_0 is the kernel of A and equal to A^dV . The sequences $\{\ker A^{i+1}\}_{i=0}^d$ and $\{A^{d-i}V\}_{i=0}^d$ both equal the flag on V induced by $\{V_i\}_{i=0}^d$. We say that A *raises* $\{V_i\}_{i=0}^d$ whenever $AV_i = V_{i+1}$ for $0 \leq i \leq d-1$ and $AV_d = 0$. Note that A raises $\{V_i\}_{i=0}^d$ if and only if A lowers the inverted decomposition $\{V_{d-i}\}_{i=0}^d$.

Lemma 2.3. *Let $\{V_i\}_{i=0}^d$ denote a decomposition of V , with idempotent sequence $\{E_i\}_{i=0}^d$. For $A \in \text{End}(V)$ the following are equivalent:*

- (i) A raises $\{V_i\}_{i=0}^d$;
- (ii) $E_i A E_j = \begin{cases} \neq 0 & \text{if } i - j = 1; \\ 0 & \text{if } i - j \neq 1 \end{cases} \quad (0 \leq i, j \leq d).$

Proof. Apply Lemma 2.2 to the decomposition $\{V_{d-i}\}_{i=0}^d$. □

Definition 2.4. An element $A \in \text{End}(V)$ will be called *Nil* whenever $A^{d+1} = 0$ and $A^d \neq 0$.

Lemma 2.5. *For $A \in \text{End}(V)$ the following are equivalent:*

- (i) A is Nil;
- (ii) there exists a decomposition of V that is lowered by A ;
- (iii) there exists a decomposition of V that is raised by A ;
- (iv) for $0 \leq i \leq d$ the kernel of A^{i+1} is $A^{d-i}V$;
- (v) the kernel of A is A^dV ;
- (vi) the sequence $\{\ker A^{i+1}\}_{i=0}^d$ is a flag on V .

Proof. (i) \Rightarrow (ii) By assumption there exists $v \in V$ such that $A^d v \neq 0$. By assumption $A^{d+1} v = 0$. Define $v_i = A^{d-i} v$ for $0 \leq i \leq d$. Then $Av_i = v_{i-1}$ for $1 \leq i \leq d$ and $Av_0 = 0$. By these comments, for $0 \leq i \leq d$ the vector v_i is in the kernel of A^{i+1} and not in the kernel of A^i . Therefore $\{v_i\}_{i=0}^d$ are linearly independent, and hence form a basis for V . By construction the induced decomposition of V is lowered by A .

(ii) \Leftrightarrow (iii) A decomposition of V is raised by A if and only if its inversion is lowered by A .

(ii) \Rightarrow (iv) By the comments above Lemma 2.3.

(iv) \Rightarrow (v) Clear.

(v) \Rightarrow (i) Observe that $A^{d+1}V = A(A^dV) = 0$, so $A^{d+1} = 0$. The map A is not invertible, so A has nonzero kernel. This kernel is A^dV , so $A^dV \neq 0$. Therefore $A^d \neq 0$. So A is Nil by Definition 2.4.

(ii) \Rightarrow (vi) By the comments above Lemma 2.3.

(vi) \Rightarrow (i) For $0 \leq i \leq d$ let U_i denote the kernel of A^{i+1} . By assumption $\{U_i\}_{i=0}^d$ is a flag on V . We have $U_d = V$, so $A^{d+1} = 0$. We have $U_{d-1} \neq V$, so $A^d \neq 0$. Therefore A is Nil by Definition 2.4. \square

We emphasize a point from Lemma 2.5. For a Nil element $A \in \text{End}(V)$ the sequence $\{A^{d-i}V\}_{i=0}^d$ is a flag on V .

3 LR pairs

In this paper, our main topic is the notion of an LR triple. As a warmup, we first consider the notion of an LR pair.

Throughout this section V denotes a vector space over \mathbb{F} with dimension $d + 1$.

Definition 3.1. An ordered pair A, B of elements in $\text{End}(V)$ is called *lowering-raising* (or *LR*) whenever there exists a decomposition of V that is lowered by A and raised by B . We refer to such a pair as an *LR pair on V* . This LR pair is said to be *over \mathbb{F}* . We call V the *underlying vector space*. We call d the *diameter* of the pair.

Lemma 3.2. Let A, B denote an LR pair on V . Then B, A is an LR pair on V .

Lemma 3.3. Let A, B denote an LR pair on V . Then each of A, B is Nil.

We mention a very special case.

Example 3.4. Assume that $d = 0$. Then $A, B \in \text{End}(V)$ form an LR pair if and only if $A = 0$ and $B = 0$. This LR pair will be called *trivial*.

Let A, B denote an LR pair on V . By Definition 3.1, there exists a decomposition $\{V_i\}_{i=0}^d$ of V that is lowered by A and raised by B . We have $V_i = A^{d-i}V_d = B^iV_0$ for $0 \leq i \leq d$. Moreover $V_0 = A^dV$ and $V_d = B^dV$. Therefore $V_i = A^{d-i}B^dV = B^iA^dV$ for $0 \leq i \leq d$. The decomposition $\{V_i\}_{i=0}^d$ is uniquely determined by A, B ; we call $\{V_i\}_{i=0}^d$ the (A, B) -decomposition of V . Its inversion $\{V_{d-i}\}_{i=0}^d$ is the (B, A) -decomposition of V .

Definition 3.5. Let A, B denote an LR pair on V . By the *idempotent sequence* for A, B we mean the idempotent sequence for the (A, B) -decomposition of V .

We have some comments.

Lemma 3.6. *Let A, B denote an LR pair on V , with idempotent sequence $\{E_i\}_{i=0}^d$. Then the LR pair B, A has idempotent sequence $\{E_{d-i}\}_{i=0}^d$.*

Lemma 3.7. *Let A, B denote an LR pair on V . The (A, B) -decomposition of V induces the flag $\{A^{d-i}V\}_{i=0}^d$. The (B, A) -decomposition of V induces the flag $\{B^{d-i}V\}_{i=0}^d$. The flags $\{A^{d-i}V\}_{i=0}^d$ and $\{B^{d-i}V\}_{i=0}^d$ are opposite.*

Lemma 3.8. *Let A, B denote an LR pair on V . For nonzero $\alpha, \beta \in \mathbb{F}$ the pair $\alpha A, \beta B$ is an LR pair on V . The $(\alpha A, \beta B)$ -decomposition of V is equal to the (A, B) -decomposition of V . Moreover the idempotent sequence for $\alpha A, \beta B$ is equal to the idempotent sequence for A, B .*

Lemma 3.9. *Let A, B denote an LR pair on V . For $0 \leq r, s \leq d$, consider the action of the map $A^r B^s$ on the (A, B) -decomposition of V . The map sends the s -component onto the $(d-r)$ -component. The map sends all other components to zero.*

Lemma 3.10. *Let A, B denote an LR pair on V . Then the following is a basis for the \mathbb{F} -vector space $\text{End}(V)$:*

$$A^r B^s \quad 0 \leq r, s \leq d. \quad (2)$$

Proof. The dimension of $\text{End}(V)$ is $(d+1)^2$. The list (2) contains $(d+1)^2$ elements, and these are linearly independent by Lemma 3.9. The result follows. \square

Corollary 3.11. *Let A, B denote an LR pair on V . Then the \mathbb{F} -algebra $\text{End}(V)$ is generated by A, B .*

Proof. By Lemma 3.10. \square

Lemma 3.12. *Let A, B denote an LR pair on V . Let $\{V_i\}_{i=0}^d$ denote the (A, B) -decomposition of V . Then the following (i)–(iv) hold.*

- (i) *For $0 \leq i \leq d$ the subspace V_i is invariant under AB and BA .*
- (ii) *The map BA is zero on V_0 .*
- (iii) *The map AB is zero on V_d .*
- (iv) *For $1 \leq i \leq d$, the eigenvalue of AB on V_{i-1} is nonzero and equal to the eigenvalue of BA on V_i .*

Proof. (i)–(iii) The decomposition $\{V_i\}_{i=0}^d$ is lowered by A and raised by B .

(iv) Pick $0 \neq u \in V_{i-1}$ and $0 \neq v \in V_i$. There exist nonzero $r, s \in \mathbb{F}$ such that $Av = ru$ and $Bu = sv$. The scalar rs is the eigenvalue of AB on V_{i-1} , and the eigenvalue of BA on V_i . \square

Definition 3.13. Let A, B denote an LR pair on V . Let $\{V_i\}_{i=0}^d$ denote the (A, B) -decomposition of V . For $1 \leq i \leq d$ let φ_i denote the eigenvalue referred to in Lemma 3.12(iv). Thus $0 \neq \varphi_i \in \mathbb{F}$. The sequence $\{\varphi_i\}_{i=1}^d$ is called the *parameter sequence* for A, B . For notational convenience define $\varphi_0 = 0$ and $\varphi_{d+1} = 0$.

Lemma 3.14. *Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. Then the LR pair B, A has parameter sequence $\{\varphi_{d-i+1}\}_{i=1}^d$.*

Proof. Use Lemma 3.12 and Definition 3.13. □

Here is an example of an LR pair.

Example 3.15. Let $\{\varphi_i\}_{i=1}^d$ denote a sequence of nonzero scalars in \mathbb{F} . Let $\{v_i\}_{i=0}^d$ denote a basis for V . Define $A \in \text{End}(V)$ such that $Av_i = \varphi_i v_{i-1}$ for $1 \leq i \leq d$ and $Av_0 = 0$. Define $B \in \text{End}(V)$ such that $Bv_i = v_{i+1}$ for $0 \leq i \leq d-1$ and $Bv_d = 0$. Then the pair A, B is an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. The (A, B) -decomposition of V is induced by the basis $\{v_i\}_{i=0}^d$.

Let A, B denote an LR pair on V , with idempotent sequence $\{E_i\}_{i=0}^d$. Our next goal is to obtain each E_i in terms of A, B .

Lemma 3.16. *Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$ and idempotent sequence $\{E_i\}_{i=0}^d$. Then*

$$AB = \sum_{j=0}^{d-1} E_j \varphi_{j+1}, \quad BA = \sum_{j=1}^d E_j \varphi_j. \quad (3)$$

Proof. Use Definitions 3.5, 3.13. □

The following result is a generalization of Lemma 3.16.

Lemma 3.17. *Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$ and idempotent sequence $\{E_i\}_{i=0}^d$. Then for $0 \leq r \leq d$,*

$$A^r B^r = \sum_{j=0}^{d-r} E_j \varphi_{j+1} \varphi_{j+2} \cdots \varphi_{j+r}, \quad (4)$$

$$B^r A^r = \sum_{j=r}^d E_j \varphi_j \varphi_{j-1} \cdots \varphi_{j-r+1}. \quad (5)$$

Proof. To verify (4), note that for $0 \leq i \leq d$, the two sides agree on component i of the (A, B) -decomposition of V . Line (5) is similarly verified. □

Lemma 3.18. *Let A, B denote an LR pair on V , with idempotent sequence $\{E_i\}_{i=0}^d$. Then for $0 \leq i \leq d$,*

$$E_i = \frac{A^{d-i} B^d A^i}{\varphi_1 \cdots \varphi_d}, \quad E_i = \frac{B^i A^d B^{d-i}}{\varphi_1 \cdots \varphi_d}, \quad (6)$$

where $\{\varphi_j\}_{j=1}^d$ is the parameter sequence for A, B .

Proof. To obtain the formula on the left in (6), in the equation $A^{d-i} B^d A^i = A^{d-i} B^{d-i} B^i A^i$, evaluate the right-hand side using Lemma 3.17 and simplify the result using $E_r E_s = \delta_{r,s} E_r$ ($0 \leq r, s \leq d$). The formula on the right in (6) is similarly obtained. □

Lemma 3.19. Let A, B denote an LR pair on V , with idempotent sequence $\{E_i\}_{i=0}^d$. Then for $0 \leq i < j \leq d$ the following are zero:

$$A^j E_i, \quad E_j A^{d-i}, \quad E_i B^j, \quad B^{d-i} E_j.$$

Proof. By (6) together with $A^{d+1} = 0$ and $B^{d+1} = 0$. \square

Lemma 3.20. Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$ and idempotent sequence $\{E_i\}_{i=0}^d$. Then for $0 \leq i \leq d$,

$$\text{tr}(ABE_i) = \varphi_{i+1}, \quad \text{tr}(BAE_i) = \varphi_i. \quad (7)$$

Proof. In the equation on the left in (3), multiply each side on the right by E_i to get $ABE_i = \varphi_{i+1}E_i$. In this equation, take the trace of each side, and recall that E_i has trace 1. This gives the equation on the left in (7). The other equation in (7) is similarly verified. \square

Let A, B denote an LR pair on V . We now describe a set of bases for V , called (A, B) -bases.

Definition 3.21. Let A, B denote an LR pair on V . Let $\{V_i\}_{i=0}^d$ denote the (A, B) -decomposition of V . A basis $\{v_i\}_{i=0}^d$ for V is called an (A, B) -basis whenever:

- (i) $v_i \in V_i$ for $0 \leq i \leq d$;
- (ii) $Av_i = v_{i-1}$ for $1 \leq i \leq d$.

Lemma 3.22. Let A, B denote an LR pair on V . Let $\{v_i\}_{i=0}^d$ denote an (A, B) -basis for V , and let $\{v'_i\}_{i=0}^d$ denote any vectors in V . Then the following are equivalent:

- (i) $\{v'_i\}_{i=0}^d$ is an (A, B) -basis for V ;
- (ii) there exists $0 \neq \zeta \in \mathbb{F}$ such that $v'_i = \zeta v_i$ for $0 \leq i \leq d$.

Proof. Use Definition 3.21. \square

Lemma 3.23. Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. Let $\{v_i\}_{i=0}^d$ denote a basis for V . Then the following are equivalent:

- (i) $\{v_i\}_{i=0}^d$ is an (A, B) -basis for V ;
- (ii) with respect to $\{v_i\}_{i=0}^d$ the matrices representing A and B are

$$A : \begin{pmatrix} 0 & 1 & & & \mathbf{0} \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & \ddots \\ \mathbf{0} & & & & 1 \\ & & & & & 0 \end{pmatrix}, \quad B : \begin{pmatrix} 0 & & & & \mathbf{0} \\ \varphi_1 & 0 & & & \\ & \varphi_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ \mathbf{0} & & & & \varphi_d & 0 \end{pmatrix}. \quad (8)$$

Proof. (i) \Rightarrow (ii) Use Definitions 3.13, 3.21.

(ii) \Rightarrow (i) Let $\{V_i\}_{i=0}^d$ denote the decomposition of V induced by $\{v_i\}_{i=0}^d$. By (8), $\{V_i\}_{i=0}^d$ is lowered by A and raised by B . Therefore $\{V_i\}_{i=0}^d$ is the (A, B) -decomposition of V . Now by Definition 3.21, $\{v_i\}_{i=0}^d$ is an (A, B) -basis for V . \square

Lemma 3.24. Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. Let $\{v_i\}_{i=0}^d$ denote any vectors in V . Then the following are equivalent:

- (i) $\{v_i\}_{i=0}^d$ is an (A, B) -basis for V ;
- (ii) $0 \neq v_0 \in A^d V$ and $Bv_i = \varphi_{i+1}v_{i+1}$ for $0 \leq i \leq d-1$;
- (iii) there exists $0 \neq \eta \in A^d V$ such that $v_i = (\varphi_1\varphi_2 \cdots \varphi_i)^{-1}B^i\eta$ for $0 \leq i \leq d$;
- (iv) $0 \neq v_d \in B^d V$ and $Av_i = v_{i-1}$ for $1 \leq i \leq d$;
- (v) there exists $0 \neq \xi \in B^d V$ such that $v_i = A^{d-i}\xi$ for $0 \leq i \leq d$.

Proof. Use Lemma 3.23. □

Let A, B denote an LR pair on V . By an *inverted* (A, B) -basis for V we mean the inversion of an (A, B) -basis for V .

Lemma 3.25. Let A, B denote an LR pair on V . Let $\{V_i\}_{i=0}^d$ denote the (A, B) -decomposition of V . A basis $\{v_i\}_{i=0}^d$ for V is an inverted (A, B) -basis if and only if both

- (i) $v_i \in V_{d-i}$ for $0 \leq i \leq d$;
- (ii) $Av_i = v_{i+1}$ for $0 \leq i \leq d-1$.

Proof. By Definition 3.21 and the meaning of inversion. □

Lemma 3.26. Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. Let $\{v_i\}_{i=0}^d$ denote a basis for V . Then the following are equivalent:

- (i) $\{v_i\}_{i=0}^d$ is an inverted (A, B) -basis for V ;
- (ii) with respect to $\{v_i\}_{i=0}^d$ the matrices representing A and B are

$$A : \begin{pmatrix} 0 & & & & \mathbf{0} \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & 0 \end{pmatrix}, \quad B : \begin{pmatrix} 0 & \varphi_d & & & \mathbf{0} \\ & 0 & \varphi_{d-1} & & \\ & & 0 & \ddots & \\ & & & \ddots & \varphi_1 \\ \mathbf{0} & & & & 0 \end{pmatrix}. \quad (9)$$

Proof. By Lemma 3.23 and the meaning of inversion. □

Lemma 3.27. Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. Let $\{v_i\}_{i=0}^d$ denote any vectors in V . Then the following are equivalent:

- (i) $\{v_i\}_{i=0}^d$ is an inverted (A, B) -basis for V ;
- (ii) $0 \neq v_0 \in B^d V$ and $Av_i = v_{i+1}$ for $0 \leq i \leq d-1$;
- (iii) there exists $0 \neq \xi \in B^d V$ such that $v_i = A^i\xi$ for $0 \leq i \leq d$;

- (iv) $0 \neq v_d \in A^d V$ and $Bv_i = \varphi_{d-i+1}v_{i-1}$ for $1 \leq i \leq d$;
- (v) there exists $0 \neq \eta \in A^d V$ such that $v_i = (\varphi_1 \varphi_2 \cdots \varphi_{d-i})^{-1} B^{d-i} \eta$ for $0 \leq i \leq d$.

Proof. By Lemma 3.24 and the meaning of inversion. \square

Let A, B denote an LR pair on V . We now consider a (B, A) -basis for V .

Lemma 3.28. *Let A, B denote an LR pair on V . Let $\{V_i\}_{i=0}^d$ denote the (A, B) -decomposition of V . A basis $\{v_i\}_{i=0}^d$ for V is a (B, A) -basis if and only if both*

- (i) $v_i \in V_{d-i}$ for $0 \leq i \leq d$;
- (ii) $Bv_i = v_{i-1}$ for $1 \leq i \leq d$.

Proof. Apply Definition 3.21 to the LR pair B, A . \square

Lemma 3.29. *Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. Let $\{v_i\}_{i=0}^d$ denote a basis for V . Then the following are equivalent:*

- (i) $\{v_i\}_{i=0}^d$ is a (B, A) -basis for V ;
- (ii) with respect to $\{v_i\}_{i=0}^d$ the matrices representing A and B are

$$A : \begin{pmatrix} 0 & & & & \mathbf{0} \\ \varphi_d & 0 & & & \\ & \varphi_{d-1} & 0 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & \varphi_1 & 0 \end{pmatrix}, \quad B : \begin{pmatrix} 0 & 1 & & & \mathbf{0} \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ \mathbf{0} & & & & 0 \end{pmatrix}. \quad (10)$$

Proof. Apply Lemma 3.23 to the LR pair B, A . \square

Lemma 3.30. *Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. Let $\{v_i\}_{i=0}^d$ denote any vectors in V . Then the following are equivalent:*

- (i) $\{v_i\}_{i=0}^d$ is a (B, A) -basis for V ;
- (ii) $0 \neq v_0 \in B^d V$ and $Av_i = \varphi_{d-i}v_{i+1}$ for $0 \leq i \leq d-1$;
- (iii) there exists $0 \neq \xi \in B^d V$ such that $v_i = (\varphi_d \varphi_{d-1} \cdots \varphi_{d-i+1})^{-1} A^i \xi$ for $0 \leq i \leq d$;
- (iv) $0 \neq v_d \in A^d V$ and $Bv_i = v_{i-1}$ for $1 \leq i \leq d$;
- (v) there exists $0 \neq \eta \in A^d V$ such that $v_i = B^{d-i} \eta$ for $0 \leq i \leq d$.

Proof. Apply Lemma 3.24 to the LR pair B, A . \square

Let A, B denote an LR pair on V . We now consider an inverted (B, A) -basis for V .

Lemma 3.31. *Let A, B denote an LR pair on V . Let $\{V_i\}_{i=0}^d$ denote the (A, B) -decomposition of V . A basis $\{v_i\}_{i=0}^d$ for V is an inverted (B, A) -basis if and only if both*

- (i) $v_i \in V_i$ for $0 \leq i \leq d$;
- (ii) $Bv_i = v_{i+1}$ for $0 \leq i \leq d-1$.

Proof. Apply Lemma 3.25 to the LR pair B, A . □

Lemma 3.32. *Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. Let $\{v_i\}_{i=0}^d$ denote a basis for V . Then the following are equivalent:*

- (i) $\{v_i\}_{i=0}^d$ is an inverted (B, A) -basis for V ;
- (ii) with respect to $\{v_i\}_{i=0}^d$ the matrices representing A and B are

$$A : \begin{pmatrix} 0 & \varphi_1 & & & \mathbf{0} \\ & 0 & \varphi_2 & & \\ & & 0 & \cdot & \\ & & & \cdot & \cdot \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & 0 \end{pmatrix}, \quad B : \begin{pmatrix} 0 & & & & \mathbf{0} \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ \mathbf{0} & & & & 1 & 0 \end{pmatrix}. \quad (11)$$

Proof. Apply Lemma 3.26 to the LR pair B, A . □

Lemma 3.33. *Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. Let $\{v_i\}_{i=0}^d$ denote any vectors in V . Then the following are equivalent:*

- (i) $\{v_i\}_{i=0}^d$ is an inverted (B, A) -basis for V ;
- (ii) $0 \neq v_0 \in A^d V$ and $Bv_i = v_{i+1}$ for $0 \leq i \leq d-1$;
- (iii) there exists $0 \neq \eta \in A^d V$ such that $v_i = B^i \eta$ for $0 \leq i \leq d$;
- (iv) $0 \neq v_d \in B^d V$ and $Av_i = \varphi_i v_{i-1}$ for $1 \leq i \leq d$;
- (v) there exists $0 \neq \xi \in B^d V$ such that $v_i = (\varphi_d \varphi_{d-1} \cdots \varphi_{i+1})^{-1} A^{d-i} \xi$ for $0 \leq i \leq d$.

Proof. Apply Lemma 3.27 to the LR pair B, A . □

Let A, B denote an LR pair on V . Earlier we used A, B to obtain four bases for V . We now consider some transitions between these bases.

Lemma 3.34. *Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$.*

- (i) *Let $\{v_i\}_{i=0}^d$ denote an (A, B) -basis for V . Then the sequence $\{\varphi_1 \varphi_2 \cdots \varphi_i v_i\}_{i=0}^d$ is an inverted (B, A) -basis for V .*
- (ii) *Let $\{v_i\}_{i=0}^d$ denote an inverted (B, A) -basis for V . Then $\{(\varphi_1 \varphi_2 \cdots \varphi_i)^{-1} v_i\}_{i=0}^d$ is an (A, B) -basis for V .*
- (iii) *Let $\{v_i\}_{i=0}^d$ denote a (B, A) -basis for V . Then the sequence $\{(\varphi_1 \varphi_2 \cdots \varphi_{d-i})^{-1} v_i\}_{i=0}^d$ is an inverted (A, B) -basis for V .*

- (iv) Let $\{v_i\}_{i=0}^d$ denote an inverted (A, B) -basis for V . Then $\{\varphi_1\varphi_2\cdots\varphi_{d-i}v_i\}_{i=0}^d$ is a (B, A) -basis for V .

Proof. (i), (ii) Compare Lemma 3.24(iii) and Lemma 3.33(iii).

(iii), (iv) Compare Lemma 3.27(v) and Lemma 3.30(v). □

We now discuss isomorphisms for LR pairs.

Definition 3.35. Let A, B denote an LR pair on V . Let V' denote a vector space over \mathbb{F} with dimension $d + 1$, and let A', B' denote an LR pair on V' . By an *isomorphism of LR pairs from A, B to A', B'* we mean an \mathbb{F} -linear bijection $\sigma : V \rightarrow V'$ such that $\sigma A = A'\sigma$ and $\sigma B = B'\sigma$. The LR pairs A, B and A', B' are called *isomorphic* whenever there exists an isomorphism of LR pairs from A, B to A', B' .

We now classify the LR pairs up to isomorphism.

Proposition 3.36. Consider the map which sends an LR pair to its parameter sequence. This map induces a bijection between the following two sets:

- (i) the isomorphism classes of LR pairs over \mathbb{F} that have diameter d ;
- (ii) the sequences $\{\varphi_i\}_{i=1}^d$ of nonzero scalars in \mathbb{F} .

Proof. By Example 3.15 and Lemma 3.32. □

We have some comments about Definition 3.35.

Lemma 3.37. Referring to Definition 3.35, let $\{E_i\}_{i=0}^d$ and $\{E'_i\}_{i=0}^d$ denote the idempotent sequences for A, B and A', B' respectively. Let σ denote an isomorphism of LR pairs from A, B to A', B' . Then $\sigma E_i = E'_i \sigma$ for $0 \leq i \leq d$.

Proof. Use Lemma 3.18. □

Lemma 3.38. Let A, B denote an LR pair on V . For nonzero $\sigma \in \text{End}(V)$ the following are equivalent:

- (i) σ is an isomorphism of LR pairs from A, B to A, B ;
- (ii) σ commutes with A and B ;
- (iii) σ commutes with everything in $\text{End}(V)$;
- (iv) there exists $0 \neq \zeta \in \mathbb{F}$ such that $\sigma = \zeta I$.

Proof. (i) \Rightarrow (ii) By Definition 3.35.

(ii) \Rightarrow (iii) By Corollary 3.11.

(iii) \Rightarrow (iv) By linear algebra.

(iv) \Rightarrow (i) By Definition 3.35. □

We have some comments about Lemma 3.8.

Lemma 3.39. Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. For nonzero $\alpha, \beta \in \mathbb{F}$ the LR pair $\alpha A, \beta B$ has parameter sequence $\{\alpha\beta\varphi_i\}_{i=1}^d$.

Proof. Use Definition 3.13. □

Lemma 3.40. Let A, B denote a nontrivial LR pair over \mathbb{F} . For nonzero $\alpha, \beta \in \mathbb{F}$ the following are equivalent:

- (i) the LR pairs A, B and $\alpha A, \beta B$ are isomorphic;
- (ii) $\alpha\beta = 1$.

Proof. Use Proposition 3.36 and Lemma 3.39. □

Lemma 3.41. Let A, B denote an LR pair on V . Pick nonzero $\alpha, \beta \in \mathbb{F}$.

- (i) Let $\{v_i\}_{i=0}^d$ denote an (A, B) -basis for V . Then the sequence $\{\alpha^{-i}v_i\}_{i=0}^d$ is an $(\alpha A, \beta B)$ -basis for V .
- (ii) Let $\{v_i\}_{i=0}^d$ denote an inverted (A, B) -basis for V . Then the sequence $\{\alpha^i v_i\}_{i=0}^d$ is an inverted $(\alpha A, \beta B)$ -basis for V .
- (iii) Let $\{v_i\}_{i=0}^d$ denote a (B, A) -basis for V . Then the sequence $\{\beta^{-i}v_i\}_{i=0}^d$ is a $(\beta B, \alpha A)$ -basis for V .
- (iv) Let $\{v_i\}_{i=0}^d$ denote an inverted (B, A) -basis for V . Then the sequence $\{\beta^i v_i\}_{i=0}^d$ is an inverted $(\beta B, \alpha A)$ -basis for V .

Proof. To obtain part (i) use Lemma 3.8 and Definition 3.21. Parts (ii)–(iv) are similarly obtained. □

Definition 3.42. Let $\{U_i\}_{i=0}^d$ denote a flag on V . An element $A \in \text{End}(V)$ is said to *lower* $\{U_i\}_{i=0}^d$ whenever $AU_i = U_{i-1}$ for $1 \leq i \leq d$ and $AU_0 = 0$. The map A is said to *raise* $\{U_i\}_{i=0}^d$ whenever $U_i + AU_i = U_{i+1}$ for $0 \leq i \leq d-1$.

Lemma 3.43. Let $\{V_i\}_{i=0}^d$ denote a decomposition of V that is lowered by $A \in \text{End}(V)$.

- (i) The flag induced by $\{V_i\}_{i=0}^d$ is lowered by A .
- (ii) The flag induced by $\{V_{d-i}\}_{i=0}^d$ is raised by A .

Proof. (i) Let $\{U_i\}_{i=0}^d$ denote the flag on V that is induced by $\{V_i\}_{i=0}^d$. Then $U_i = V_0 + \cdots + V_i$ for $0 \leq i \leq d$. By assumption $AV_i = V_{i-1}$ for $1 \leq i \leq d$ and $AV_0 = 0$. Therefore $AU_i = U_{i-1}$ for $1 \leq i \leq d$ and $AU_0 = 0$. In other words, the flag $\{U_i\}_{i=0}^d$ is lowered by A .

(ii) Let $\{U_i\}_{i=0}^d$ denote the flag on V that is induced by $\{V_{d-i}\}_{i=0}^d$. Then $U_i = V_{d-i} + \cdots + V_d$ for $0 \leq i \leq d$. For $0 \leq i \leq d-1$, $AU_i = V_{d-i-1} + \cdots + V_{d-1}$. By these comments $U_i + AU_i = U_{i+1}$. Therefore $\{U_i\}_{i=0}^d$ is raised by A . □

Lemma 3.44. Let A, B denote an LR pair on V .

- (i) The flag $\{A^{d-i}V\}_{i=0}^d$ is lowered by A and raised by B .

(ii) The flag $\{B^{d-i}V\}_{i=0}^d$ is raised by A and lowered by B .

Proof. The (B, A) -decomposition of V is the inversion of the (A, B) -decomposition of V . The (A, B) -decomposition of V is lowered by A . The (B, A) -decomposition of V is lowered by B . By Lemma 3.7, the (A, B) -decomposition of V induces the flag $\{A^{d-i}V\}_{i=0}^d$, and the (B, A) -decomposition of V induces the flag $\{B^{d-i}V\}_{i=0}^d$. The result follows in view of Lemma 3.43. \square

Lemma 3.45. Let $\{U_i\}_{i=0}^d$ and $\{U'_i\}_{i=0}^d$ denote flags on V . Then the following are equivalent:

(i) $\{U_i\}_{i=0}^d$ and $\{U'_i\}_{i=0}^d$ are opposite;

(ii) there exists $A \in \text{End}(V)$ that lowers $\{U_i\}_{i=0}^d$ and raises $\{U'_i\}_{i=0}^d$.

Assume that (i), (ii) hold, and define $V_i = U_i \cap U'_{d-i}$ for $0 \leq i \leq d$. Then the decomposition $\{V_i\}_{i=0}^d$ is lowered by A .

Proof. (i) \Rightarrow (ii) Define $V_i = U_i \cap U'_{d-i}$ for $0 \leq i \leq d$. Then $\{V_i\}_{i=0}^d$ is a decomposition of V that induces $\{U_i\}_{i=0}^d$ and whose inversion induces $\{U'_i\}_{i=0}^d$. Let $A \in \text{End}(V)$ lower $\{V_i\}_{i=0}^d$. By Lemma 3.43, A lowers $\{U_i\}_{i=0}^d$ and raises $\{U'_i\}_{i=0}^d$.

(ii) \Rightarrow (i) We display a decomposition $\{W_i\}_{i=0}^d$ of V that induces $\{U_i\}_{i=0}^d$ and whose inversion induces $\{U'_i\}_{i=0}^d$. Define $W_i = A^{d-i}U'_0$ for $0 \leq i \leq d$. We show that $\{W_i\}_{i=0}^d$ is a decomposition of V . Note that U'_0 has dimension one, so W_i has dimension at most one for $0 \leq i \leq d$. Using the assumption that A raises $\{U'_i\}_{i=0}^d$, we obtain $U'_j = W_d + W_{d-1} + \cdots + W_{d-j}$ for $0 \leq j \leq d$. Setting $j = d$ we find $V = \sum_{i=0}^d W_i$. The dimension of V is $d + 1$. Therefore the sum $V = \sum_{i=0}^d W_i$ is direct, and W_i has dimension one for $0 \leq i \leq d$. In other words $\{W_i\}_{i=0}^d$ is a decomposition of V . By construction, the inverted decomposition $\{W_{d-i}\}_{i=0}^d$ induces $\{U'_i\}_{i=0}^d$. By assumption A lowers the flag $\{U_i\}_{i=0}^d$. By Definition 3.42, $\{U_i\}_{i=0}^d$ is the unique flag on V that is lowered by A . By the definition of $\{W_i\}_{i=0}^d$ we find $AW_i = W_{i-1}$ for $1 \leq i \leq d$. Also $AW_0 = A^{d+1}V = 0$ since A is Nil. Consequently A lowers $\{W_i\}_{i=0}^d$. By Lemma 3.43, A lowers the flag induced by $\{W_i\}_{i=0}^d$. By these comments $\{W_i\}_{i=0}^d$ induces $\{U_i\}_{i=0}^d$. We have shown that the decomposition $\{W_i\}_{i=0}^d$ induces $\{U_i\}_{i=0}^d$ and the inverted decomposition $\{W_{d-i}\}_{i=0}^d$ induces $\{U'_i\}_{i=0}^d$. Therefore $\{U_i\}_{i=0}^d$ and $\{U'_i\}_{i=0}^d$ are opposite.

Assume that (i), (ii) hold. Recall from the proof of (ii) \Rightarrow (i) that the decomposition $\{W_i\}_{i=0}^d$ induces $\{U_i\}_{i=0}^d$ and the inverted decomposition $\{W_{d-i}\}_{i=0}^d$ induces $\{U'_i\}_{i=0}^d$. Therefore $W_i = U_i \cap U'_{d-i} = V_i$ for $0 \leq i \leq d$. Recall also from the proof of (ii) \Rightarrow (i) that A lowers $\{W_i\}_{i=0}^d$. So A lowers $\{V_i\}_{i=0}^d$. \square

Proposition 3.46. Let $A, B \in \text{End}(V)$. Then A, B is an LR pair on V if and only if the following (i)–(iii) hold:

(i) A and B are Nil;

(ii) the flag $\{A^{d-i}V\}_{i=0}^d$ is raised by B ;

(iii) the flag $\{B^{d-i}V\}_{i=0}^d$ is raised by A .

Proof. First assume that A, B is an LR pair on V . Then condition (i) holds by Lemma 3.3, and conditions (ii), (iii) hold by Lemma 3.44. Conversely, assume that the conditions (i)–(iii) hold. To show that A, B is an LR pair on V , we display a decomposition $\{V_i\}_{i=0}^d$ of V that is lowered by A and raised by B . By construction and since A is Nil, the flag $\{A^{d-i}V\}_{i=0}^d$ is lowered by A . By assumption the flag $\{B^{d-i}V\}_{i=0}^d$ is raised by A . Now by Lemma 3.45 the flags $\{A^{d-i}V\}_{i=0}^d$ and $\{B^{d-i}V\}_{i=0}^d$ are opposite. Define $V_i = A^{d-i}V \cap B^iV$ for $0 \leq i \leq d$. By Lemma 3.45 the decomposition $\{V_i\}_{i=0}^d$ is lowered by A . Interchanging the roles of A, B in the argument so far, we see that $\{V_i\}_{i=0}^d$ is raised by B . We have shown that A, B is an LR pair on V . \square

We define some matrices for later use.

Definition 3.47. Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. Define a diagonal matrix $D \in \text{Mat}_{d+1}(\mathbb{F})$ with (i, i) -entry $\varphi_1\varphi_2 \cdots \varphi_i$ for $0 \leq i \leq d$.

Lemma 3.48. Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. For $M \in \text{Mat}_{d+1}(\mathbb{F})$ the following are equivalent:

- (i) M is the transition matrix from an (A, B) -basis of V to an inverted (B, A) -basis of V ;
- (ii) there exists $0 \neq \zeta \in \mathbb{F}$ such that $M = \zeta D$.

Proof. Use Lemma 3.34(i). \square

Definition 3.49. Let τ denote the matrix in $\text{Mat}_{d+1}(\mathbb{F})$ that has $(i-1, i)$ -entry 1 for $1 \leq i \leq d$, and all other entries 0. Thus

$$\tau = \begin{pmatrix} 0 & 1 & & & \mathbf{0} \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ \mathbf{0} & & & & 0 \end{pmatrix}, \quad \mathbf{Z}\tau\mathbf{Z} = \begin{pmatrix} 0 & & & & \mathbf{0} \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & 0 \end{pmatrix}.$$

Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. In lines (8), (9), (10), (11) we encountered some matrices that had $\{\varphi_i\}_{i=1}^d$ among the entries. We now express these matrices in terms of \mathbf{Z} , D , τ .

Lemma 3.50. Referring to Definitions 3.47 and 3.49,

$$\begin{aligned} D^{-1}\tau D &= \begin{pmatrix} 0 & \varphi_1 & & & \mathbf{0} \\ & 0 & \varphi_2 & & \\ & & 0 & \ddots & \\ & & & \ddots & \varphi_d \\ \mathbf{0} & & & & 0 \end{pmatrix}, & \mathbf{Z}D\mathbf{Z}\tau\mathbf{Z}D^{-1}\mathbf{Z} &= \begin{pmatrix} 0 & \varphi_d & & & \mathbf{0} \\ & 0 & \varphi_{d-1} & & \\ & & 0 & \ddots & \\ & & & \ddots & \varphi_1 \\ \mathbf{0} & & & & 0 \end{pmatrix}, \\ \mathbf{Z}D^{-1}\tau D\mathbf{Z} &= \begin{pmatrix} 0 & & & & \mathbf{0} \\ \varphi_d & 0 & & & \\ & \varphi_{d-1} & 0 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & \varphi_1 & 0 \end{pmatrix}, & D\mathbf{Z}\tau\mathbf{Z}D^{-1} &= \begin{pmatrix} 0 & & & & \mathbf{0} \\ \varphi_1 & 0 & & & \\ & \varphi_2 & 0 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & \varphi_d & 0 \end{pmatrix}. \end{aligned}$$

Proof. Matrix multiplication. □

In Section 12 we will consider some powers of the matrix τ from Definition 3.49. We now compute the entries of these powers.

Lemma 3.51. *Referring to Definition 3.49, for $0 \leq r \leq d$ the matrix τ^r has (i, j) -entry*

$$(\tau^r)_{i,j} = \begin{cases} 1 & \text{if } j - i = r; \\ 0 & \text{if } j - i \neq r \end{cases} \quad (0 \leq i, j \leq d).$$

Moreover $\tau^{d+1} = 0$.

Proof. Matrix multiplication. □

Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. As we proceed, we will encounter the case in which the $\{\varphi_i\}_{i=1}^d$ satisfy a linear recurrence. We now consider this case.

Lemma 3.52. *Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. Pick an integer r ($1 \leq r \leq d+1$). Let x and $\{y_i\}_{i=0}^r$ denote scalars in \mathbb{F} . Then the following are equivalent:*

- (i) $xA^{r-1} = \sum_{i=0}^r y_i A^i B A^{r-i};$
- (ii) $xB^{r-1} = \sum_{i=0}^r y_i B^{r-i} A B^i;$
- (iii) $x = y_0 \varphi_i + y_1 \varphi_{i+1} + \cdots + y_r \varphi_{i+r}$ for $0 \leq i \leq d - r + 1$.

Proof. Represent A and B by matrices as in (8). □

4 LR pairs of Weyl and q -Weyl type

In this section we investigate two families of LR pairs, said to have Weyl type and q -Weyl type. We begin with an example that illustrates Lemma 3.52. Throughout this section V denotes a vector space over \mathbb{F} with dimension $d+1$.

Example 4.1. Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. By Lemma 3.52,

$$AB - BA = I \tag{12}$$

if and only if

$$\varphi_{i+1} - \varphi_i = 1 \quad (0 \leq i \leq d). \tag{13}$$

Note 4.2. The equation (12) is called the *Weyl relation*.

Definition 4.3. The LR pair A, B in Example 4.1 is said to have *Weyl type* whenever it satisfies the equivalent conditions (12), (13).

Note 4.4. Referring to Example 4.1, assume that A, B has Weyl type. Then the LR pair $B, -A$ has Weyl type.

Lemma 4.5. Referring to Example 4.1, assume that A, B has Weyl type. Then (i), (ii) hold below.

- (i) $\varphi_i = i$ for $1 \leq i \leq d$.
- (ii) The integer $p = d + 1$ is prime and $\text{Char}(\mathbb{F}) = p$.

Proof. By (13) and $\varphi_0 = 0$ we obtain $\varphi_i = i$ for $1 \leq i \leq d + 1$. We have $\varphi_{d+1} = 0$, so $d + 1 = 0$ in the field \mathbb{F} . For $1 \leq i \leq d$ we have $\varphi_i \neq 0$, so $i \neq 0$ in \mathbb{F} . The results follow. \square

Lemma 4.6. Assume that $p = d + 1$ is prime and $\text{Char}(\mathbb{F}) = p$. Define $\varphi_i = i$ for $1 \leq i \leq d$. Then $\{\varphi_i\}_{i=1}^d$ are nonzero; let A, B denote the LR pair over \mathbb{F} that has parameter sequence $\{\varphi_i\}_{i=1}^d$. Then A, B has Weyl type.

Proof. One checks that condition (13) is satisfied. \square

Lemma 4.7. Assume that $p = d + 1$ is prime and $\text{Char}(\mathbb{F}) = p$. Then for $A, B \in \text{End}(V)$ the following are equivalent:

- (i) neither of A, B is invertible and $AB - BA = I$;
- (ii) A, B is an LR pair on V that has Weyl type.

Proof. (i) \Rightarrow (ii) Since A is not invertible, there exists $0 \neq \eta \in V$ such that $A\eta = 0$. Define $v_i = B^i\eta$ for $0 \leq i \leq d$. By construction, $Av_0 = 0$ and $Bv_{i-1} = v_i$ for $1 \leq i \leq d$. Using $AB - BA = I$ and induction on i , we obtain $Av_i = iv_{i-1}$ for $1 \leq i \leq d$. Note that $i \neq 0$ in \mathbb{F} for $1 \leq i \leq d$. By these comments, for $0 \leq i \leq d$ the vector v_i is in the kernel of A^{i+1} and not in the kernel of A^i . Therefore $\{v_i\}_{i=0}^d$ are linearly independent and hence form a basis for V . By construction the induced decomposition $\{V_i\}_{i=0}^d$ is lowered by A . Therefore A is Nil. Replacing A, B by $B, -A$ in the above argument, we see that B is Nil. Now $Bv_d = B^{d+1}\eta = 0$. Now by construction B raises the decomposition $\{V_i\}_{i=0}^d$. We have shown that the decomposition $\{V_i\}_{i=0}^d$ is lowered by A and raised by B . Therefore A, B is an LR pair on V . This LR pair has Weyl type by Definition 4.3 and since $AB - BA = I$.

(ii) \Rightarrow (i) The elements A, B are not invertible, since they are Nil by Lemma 3.3. By Definition 4.3 we have $AB - BA = I$. \square

Later in the paper we will use the following curious fact about LR pairs of Weyl type.

Lemma 4.8. Assume $d \geq 2$. Let A, B denote an LR pair on V that has Weyl type. Define $C = -A - B$. Then the pairs B, C and C, A are LR pairs on V that have Weyl type.

Proof. By Definition 4.3, $AB - BA = I$. By Lemma 4.5(ii), $p = d + 1$ is prime and $\text{Char}(\mathbb{F}) = p$. We show that B, C is an LR pair on V that has Weyl type. To do this we apply Lemma 4.7 to the pair B, C . Using $AB - BA = I$ and the definition of C , we find $BC - CB = I$. The map B is not invertible since B is Nil. We show that C is not

invertible. By Lemma 3.32, with respect to an inverted (B, A) -basis for V the element $A + B$ is represented by

$$\begin{pmatrix} 0 & 1 & & & \mathbf{0} \\ 1 & 0 & 2 & & \\ & 1 & 0 & \cdot & \\ & & \cdot & \cdot & \cdot \\ \mathbf{0} & & & \cdot & \cdot & d \\ & & & & 1 & 0 \end{pmatrix}. \quad (14)$$

By our assumption $d \geq 2$, the prime $p = d + 1$ is odd. Therefore d is even. Define

$$z_{2i} = \frac{(-1)^i}{2^i i!} \quad (0 \leq i \leq d/2)$$

and $z_{2i+1} = 0$ for $0 \leq i < d/2$. The matrix (14) times the column vector $(z_0, z_1, \dots, z_d)^t$ is zero. Therefore the matrix (14) is not invertible. Therefore $A + B$ is not invertible, so C is not invertible. Applying Lemma 4.7 to the pair B, C we find that B, C is an LR pair on V that has Weyl type. One similarly shows that C, A is an LR pair on V that has Weyl type. \square

Here is another example of an LR pair.

Example 4.9. Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. Pick a nonzero $q \in \mathbb{F}$ such that $q^2 \neq 1$. By Lemma 3.52,

$$\frac{qAB - q^{-1}BA}{q - q^{-1}} = I \quad (15)$$

if and only if

$$\frac{q\varphi_{i+1} - q^{-1}\varphi_i}{q - q^{-1}} = 1 \quad (0 \leq i \leq d). \quad (16)$$

Note 4.10. The equation (15) is called the *q-Weyl relation*.

Definition 4.11. The LR pair A, B in Example 4.9 is said to have *q-Weyl type* whenever it satisfies the equivalent conditions (15), (16).

Note 4.12. Referring to Example 4.9, assume that A, B has *q-Weyl type*. Then A, B has $(-q)$ -Weyl type. Moreover the LR pair B, A has (q^{-1}) -Weyl type.

Lemma 4.13. Referring to Example 4.9, assume that A, B has *q-Weyl type*. Then (i)–(v) hold below.

- (i) $d \geq 1$.
- (ii) $\varphi_i = 1 - q^{-2i}$ for $1 \leq i \leq d$.
- (iii) Assume that $\text{Char}(\mathbb{F}) \neq 2$ and d is odd. Then q is a primitive $(2d + 2)$ -root of unity.

- (iv) Assume that $\text{Char}(\mathbb{F}) \neq 2$ and d is even. Then q becomes a primitive $(2d+2)$ -root of unity, after replacing q by $-q$ if necessary.
- (v) Assume that $\text{Char}(\mathbb{F}) = 2$. Then d is even. Moreover q is a primitive $(d+1)$ -root of unity.

Proof. By (16) and $\varphi_0 = 0$ along with induction on i , we obtain $\varphi_i = 1 - q^{-2i}$ for $1 \leq i \leq d+1$. We have $\varphi_{d+1} = 0$, so $q^{2d+2} = 1$. For $1 \leq i \leq d$ we have $\varphi_i \neq 0$, so $q^{2i} \neq 1$. The results follow. \square

Definition 4.14. For $q \in \mathbb{F}$ the ordered pair d, q will be called *standard* whenever the following (i)–(iii) hold.

- (i) $d \geq 1$.
- (ii) Assume $\text{Char}(\mathbb{F}) \neq 2$. Then q is a primitive $(2d+2)$ -root of unity.
- (iii) Assume $\text{Char}(\mathbb{F}) = 2$. Then d is even, and q is a primitive $(d+1)$ -root of unity.

Note 4.15. Referring to Definition 4.14, assume that d, q is standard. Then q is nonzero and $q^2 \neq 1$.

Lemma 4.16. Referring to Example 4.9, assume that A, B has q -Weyl type. Then d, q is standard or $d, -q$ is standard.

Proof. Use Lemma 4.13 and Definition 4.14. \square

For the rest of this section, the following assumption is in effect.

Assumption 4.17. Fix $q \in \mathbb{F}$ and assume that d, q is standard. We fix a square root $q^{1/2}$ in the algebraic closure $\overline{\mathbb{F}}$.

Lemma 4.18. With reference to Assumption 4.17, define $\varphi_i = 1 - q^{-2i}$ for $1 \leq i \leq d$. Then $\{\varphi_i\}_{i=1}^d$ are nonzero; let A, B denote the LR pair over \mathbb{F} that has parameter sequence $\{\varphi_i\}_{i=1}^d$. Then A, B has q -Weyl type.

Proof. Condition (16) is readily checked. \square

Lemma 4.19. With reference to Assumption 4.17, for $A, B \in \text{End}(V)$ the following are equivalent:

- (i) neither of A, B is invertible and

$$\frac{qAB - q^{-1}BA}{q - q^{-1}} = I; \tag{17}$$

- (ii) A, B is an LR pair on V that has q -Weyl type.

Proof. The proof is similar to the proof of Lemma 4.7. For the sake of completeness we give the details.

(i) \Rightarrow (ii) For $1 \leq i \leq d$ define $\varphi_i = 1 - q^{-2i}$ and note that $\varphi_i \neq 0$. Since A is not invertible, there exists $0 \neq \eta \in V$ such that $A\eta = 0$. Define $v_i = B^i\eta$ for $0 \leq i \leq d$. By construction, $Av_0 = 0$ and $Bv_{i-1} = v_i$ for $1 \leq i \leq d$. Using (17) and induction on i , we obtain $Av_i = \varphi_i v_{i-1}$ for $1 \leq i \leq d$. By these comments, for $0 \leq i \leq d$ the vector v_i is in the kernel of A^{i+1} and not in the kernel of A^i . Therefore $\{v_i\}_{i=0}^d$ are linearly independent and hence form a basis for V . By construction the induced decomposition $\{V_i\}_{i=0}^d$ is lowered by A . Therefore A is Nil. Replacing A, B, q by B, A, q^{-1} in the above argument, we see that B is Nil. Now $Bv_d = B^{d+1}\eta = 0$. Now by construction B raises the decomposition $\{V_i\}_{i=0}^d$. We have shown that the decomposition $\{V_i\}_{i=0}^d$ is lowered by A and raised by B . Therefore A, B is an LR pair on V . This LR pair has q -Weyl type by Definition 4.11 and (17).

(ii) \Rightarrow (i) The elements A, B are not invertible since they are Nil by Lemma 3.3. By Definition 4.11 the pair A, B satisfies (17). \square

With reference to Assumption 4.17, let A, B denote an LR pair on V that has q -Weyl type. Later in the paper we will need the eigenvalues of $qA + q^{-1}B$. Our next goal is to compute these eigenvalues.

Lemma 4.20. *Pick a nonzero $b \in \mathbb{F}$ such that $b^i \neq 1$ for $1 \leq i \leq d$. For the tridiagonal matrix*

$$\begin{pmatrix} 0 & b^d - 1 & & & & & \mathbf{0} \\ b - 1 & 0 & b^d - b & & & & \\ & b^2 - 1 & 0 & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & b^d - b^{d-1} & \\ \mathbf{0} & & & & b^d - 1 & 0 & \end{pmatrix} \quad (18)$$

the roots of the characteristic polynomial are

$$b^j - b^{d-j} \quad j = 0, 1, \dots, d. \quad (19)$$

For $0 \leq j \leq d$ we give a (column) eigenvector for the matrix (18) and eigenvalue $b^j - b^{d-j}$. This eigenvector has i th coordinate

$${}_3\phi_2 \left(\begin{matrix} b^{-i}, b^{j-d}, -b^{-j} \\ 0, b^{-d} \end{matrix} \middle| b, b \right)$$

for $0 \leq i \leq d$. We follow the standard notation for basic hypergeometric series [10, p. 4].

Proof. See [20, Example 5.9]. \square

Definition 4.21. With reference to Assumption 4.17, define

$$\theta_j = q^{j+1/2} + q^{-j-1/2} \quad (0 \leq j \leq d). \quad (20)$$

Lemma 4.22. *With reference to Assumption 4.17 and Definition 4.21, the following (i)–(iv) hold.*

- (i) $\theta_j = -\theta_{d-j}$ for $0 \leq j \leq d$.
- (ii) Assume that $d = 2m$ is even. Then $\theta_m = 0$.
- (iii) Assume that $\text{Char}(\mathbb{F}) \neq 2$. Then $\{\theta_j\}_{j=0}^d$ are mutually distinct.
- (iv) Assume that $\text{Char}(\mathbb{F}) = 2$, so that $d = 2m$ is even. Then $\{\theta_j\}_{j=0}^m$ are mutually distinct.

Proof. Use Definition 4.21 and the restrictions on q given in Definition 4.14. \square

Lemma 4.23. *With reference to Assumption 4.17, let A, B denote an LR pair on V that has q -Weyl type. Then for $qA + q^{-1}B$ the roots of the characteristic polynomial are $\{\theta_j\}_{j=0}^d$.*

Proof. Let H denote the matrix in $\text{Mat}_{d+1}(\mathbb{F})$ that represents $qA + q^{-1}B$ with respect to an inverted (B, A) -basis for V . The entries of H are obtained using Lemma 3.32. Let Δ denote the matrix (18), with $b = q^{-1}$. For $1 \leq i \leq d$ define $n_i = q^{1/2}(q^{-i} - 1)$ and note that $n_i \neq 0$. Define a diagonal matrix $N \in \text{Mat}_{d+1}(\overline{\mathbb{F}})$ with (i, i) -entry $n_1 n_2 \cdots n_i$ for $0 \leq i \leq d$. Note that N is invertible. By matrix multiplication $H = q^{-1/2} N^{-1} \Delta N$. One checks that

$$\theta_j = q^{-1/2}(b^j - b^{d-j}) \quad (0 \leq j \leq d).$$

By these comments and Lemma 4.20, for the matrix H the roots of the characteristic polynomial are $\{\theta_j\}_{j=0}^d$. The result follows. \square

5 The dual space V^*

Recall our vector space V over \mathbb{F} with dimension $d + 1$. Let V^* denote the vector space over \mathbb{F} consisting of the \mathbb{F} -linear maps from V to \mathbb{F} . We call V^* the *dual space* for V . The vector spaces V and V^* have the same dimension $d + 1$. There exists a bilinear form $(,) : V \times V^* \rightarrow \mathbb{F}$ such that $(u, f) = f(u)$ for all $u \in V$ and $f \in V^*$. This bilinear form is nondegenerate in the sense of [16, Section 11]. We view $(V^*)^* = V$. Nonempty subsets $X \subseteq V$ and $Y \subseteq V^*$ are called *orthogonal* whenever $(x, y) = 0$ for all $x \in X$ and $y \in Y$. For a subspace U of V (resp. V^*) let U^\perp denote the set of vectors in V^* (resp. V) that are orthogonal to everything in U . The subspace U^\perp is called the *orthogonal complement* of U . Note that $\dim(U) + \dim(U^\perp) = d + 1$.

A basis $\{v_i\}_{i=0}^d$ of V and a basis $\{v'_i\}_{i=0}^d$ of V^* are called *dual* whenever $(v_i, v'_j) = \delta_{i,j}$ for $0 \leq i, j \leq d$. Each basis of V (resp. V^*) is dual to a unique basis of V^* (resp. V). Let $\{u_i\}_{i=0}^d$ (resp. $\{v_i\}_{i=0}^d$) denote a basis of V , and let $\{u'_i\}_{i=0}^d$ (resp. $\{v'_i\}_{i=0}^d$) denote the dual basis of V^* . Then the following matrices are transpose: (i) the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$; (ii) the transition matrix from $\{v'_i\}_{i=0}^d$ to $\{u'_i\}_{i=0}^d$.

A decomposition $\{V_i\}_{i=0}^d$ of V and a decomposition $\{V'_i\}_{i=0}^d$ of V^* are called *dual* whenever V_i, V'_j are orthogonal for all i, j ($0 \leq i, j \leq d$) such that $i \neq j$. Let $\{v_i\}_{i=0}^d$ denote a basis of V and let $\{v'_i\}_{i=0}^d$ denote the dual basis of V^* . Then the following are dual: (i) the decomposition of V induced by $\{v_i\}_{i=0}^d$; (ii) the decomposition of V^* induced by $\{v'_i\}_{i=0}^d$. Each decomposition of V (resp. V^*) is dual to a unique decomposition of V^* (resp. V).

A flag $\{U_i\}_{i=0}^d$ on V and a flag $\{U'_i\}_{i=0}^d$ on V^* are called *dual* whenever U_i, U'_j are orthogonal for all i, j ($0 \leq i, j \leq d$) such that $i+j = d-1$. In this case U_i, U'_j are orthogonal complements for all i, j ($0 \leq i, j \leq d$) such that $i+j = d-1$. Each flag on V (resp. V^*) is dual to a unique flag on V^* (resp. V).

Lemma 5.1. *Let $\{V_i\}_{i=0}^d$ denote a decomposition of V and let $\{V'_i\}_{i=0}^d$ denote the dual decomposition of V^* . Then the following are dual:*

- (i) *the flag on V induced by $\{V_i\}_{i=0}^d$;*
- (ii) *the flag on V^* induced by $\{V'_{d-i}\}_{i=0}^d$.*

Proof. Let $\{U_i\}_{i=0}^d$ and $\{U'_i\}_{i=0}^d$ denote the flags from (i) and (ii), respectively. For $0 \leq i \leq d$ we have $U_i = V_0 + \cdots + V_i$ and $U'_i = V'_{d-i} + \cdots + V'_d$. For $0 \leq i, j \leq d$ such that $i+j = d-1$, the subspace $U'_j = V'_{i+1} + \cdots + V'_d$ is orthogonal to U_i . The result follows. \square

For \mathbb{F} -algebras \mathcal{A} and \mathcal{A}' , a map $\sigma : \mathcal{A} \rightarrow \mathcal{A}'$ is called an \mathbb{F} -algebra *antiisomorphism* whenever σ is an isomorphism of \mathbb{F} -vector spaces and $(ab)^\sigma = b^\sigma a^\sigma$ for all $a, b \in \mathcal{A}$. By an *antiautomorphism* of \mathcal{A} we mean an \mathbb{F} -algebra antiisomorphism $\sigma : \mathcal{A} \rightarrow \mathcal{A}$. For $X \in \text{End}(V)$ there exists a unique element of $\text{End}(V^*)$, denoted \tilde{X} , such that $(Xu, v) = (u, \tilde{X}v)$ for all $u \in V$ and $v \in V^*$. The map \tilde{X} is called the *adjoint* of X . The adjoint map $\text{End}(V) \rightarrow \text{End}(V^*)$, $X \mapsto \tilde{X}$ is an \mathbb{F} -algebra antiisomorphism. Let $\{v_i\}_{i=0}^d$ denote a basis of V and let $\{v'_i\}_{i=0}^d$ denote the dual basis of V^* . Then for $X \in \text{End}(V)$ the following matrices are transpose: (i) the matrix representing X with respect to $\{v_i\}_{i=0}^d$; (ii) the matrix representing \tilde{X} with respect to $\{v'_i\}_{i=0}^d$.

Let $\{V_i\}_{i=0}^d$ denote a decomposition of V and let $\{V'_i\}_{i=0}^d$ denote the dual decomposition of V^* . Let $\{E_i\}_{i=0}^d$ denote the idempotent sequence for $\{V_i\}_{i=0}^d$. Then $\{\tilde{E}_i\}_{i=0}^d$ is the idempotent sequence for $\{V'_i\}_{i=0}^d$.

Lemma 5.2. *Let $\{V_i\}_{i=0}^d$ denote a decomposition of V and let $\{V'_i\}_{i=0}^d$ denote the dual decomposition of V^* . Then for $A \in \text{End}(V)$,*

- (i) *A lowers $\{V_i\}_{i=0}^d$ if and only if \tilde{A} raises $\{V'_i\}_{i=0}^d$;*
- (ii) *A raises $\{V_i\}_{i=0}^d$ if and only if \tilde{A} lowers $\{V'_i\}_{i=0}^d$.*

Proof. (i) We invoke Lemmas 2.2, 2.3. Let $\{E_i\}_{i=0}^d$ denote the idempotent sequence for $\{V_i\}_{i=0}^d$. Then $\{\tilde{E}_i\}_{i=0}^d$ is the idempotent sequence for $\{V'_i\}_{i=0}^d$. Recall that the adjoint map is an antiisomorphism. So for $0 \leq i, j \leq d$, $E_i A E_j = 0$ if and only if $\tilde{E}_j \tilde{A} \tilde{E}_i = 0$. Consequently Lemma 2.2(ii) holds for $\{E_i\}_{i=0}^d$ and A , if and only if Lemma 2.3(ii) holds for $\{\tilde{E}_i\}_{i=0}^d$ and \tilde{A} . The result now follows in view of Lemmas 2.2, 2.3.

(ii) Similar to the proof of (i) above. \square

Lemma 5.3. *For $A \in \text{End}(V)$, A is Nil if and only if \tilde{A} is Nil. In this case the following flags are dual:*

$$\{A^{d-i}V\}_{i=0}^d, \quad \{\tilde{A}^{d-i}V^*\}_{i=0}^d. \quad (21)$$

Proof. The adjoint map is an antiisomorphism. So for $0 \leq i \leq d+1$, $A^i = 0$ if and only if $\tilde{A}^i = 0$. Therefore, A is Nil if and only if \tilde{A} is Nil. In this case, the flags (21) are dual since for $0 \leq i, j \leq d$ such that $i + j = d - 1$,

$$(A^{d-i}V, \tilde{A}^{d-j}V^*) = (A^{d-j}A^{d-i}V, V^*) = (A^{d+1}V, V^*) = (0, V^*) = 0.$$

□

Lemma 5.4. *Let A, B denote an LR pair on V . Then the following (i)–(iii) hold.*

- (i) *The pair \tilde{A}, \tilde{B} is an LR pair on V^* .*
- (ii) *The (A, B) -decomposition of V is dual to the (\tilde{B}, \tilde{A}) -decomposition of V^* .*
- (iii) *The (B, A) -decomposition of V is dual to the (\tilde{A}, \tilde{B}) -decomposition of V^* .*

Proof. Let $\{V_i\}_{i=0}^d$ denote the (A, B) -decomposition of V . Let $\{V'_i\}_{i=0}^d$ denote the dual decomposition of V^* . By construction $\{V_i\}_{i=0}^d$ is lowered by A and raised by B . By Lemma 5.2, $\{V'_i\}_{i=0}^d$ is raised by \tilde{A} and lowered by \tilde{B} . Therefore, the decomposition $\{V'_{d-i}\}_{i=0}^d$ is lowered by \tilde{A} and raised by \tilde{B} . The results follow. □

Lemma 5.5. *Let A, B denote an LR pair on V , with idempotent sequence $\{E_i\}_{i=0}^d$. Then the LR pair \tilde{A}, \tilde{B} has idempotent sequence $\{\tilde{E}_{d-i}\}_{i=0}^d$.*

Proof. By the assertion above Lemma 5.2, along with Lemma 5.4(iii). □

Lemma 5.6. *Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$. Then the LR pair \tilde{A}, \tilde{B} has parameter sequence $\{\varphi_{d-i+1}\}_{i=1}^d$.*

Proof. An element in $\text{End}(V)$ has the same trace as its adjoint. Let $\{\tilde{\varphi}_i\}_{i=1}^d$ denote the parameter sequence for \tilde{A}, \tilde{B} . For $1 \leq i \leq d$ we show that $\tilde{\varphi}_i = \varphi_{d-i+1}$. By Lemmas 3.20, 5.5 and since $\text{tr}(XY) = \text{tr}(YX)$,

$$\tilde{\varphi}_i = \text{tr}(\tilde{B}\tilde{A}\tilde{E}_{d-i}) = \text{tr}(E_{d-i}AB) = \text{tr}(ABE_{d-i}) = \varphi_{d-i+1}.$$

The result follows. □

Lemma 5.7. *Let A, B denote an LR pair on V . Then the following (i)–(iv) hold.*

- (i) *For an (A, B) -basis of V , its dual is an inverted (\tilde{A}, \tilde{B}) -basis of V^* .*
- (ii) *For an inverted (A, B) -basis of V , its dual is a (\tilde{A}, \tilde{B}) -basis of V^* .*
- (iii) *For a (B, A) -basis of V , its dual is an inverted (\tilde{B}, \tilde{A}) -basis of V^* .*
- (iv) *For an inverted (B, A) -basis of V , its dual is a (\tilde{B}, \tilde{A}) -basis of V^* .*

Proof. (i) Let $\{v_i\}_{i=0}^d$ denote an (A, B) -basis of V . Let $\{v'_i\}_{i=0}^d$ denote the dual basis of V^* . With respect to $\{v_i\}_{i=0}^d$ the matrices representing A, B are given in (8). For these matrices the transpose represents \tilde{A}, \tilde{B} with respect to $\{v'_i\}_{i=0}^d$. Applying Lemma 3.26 to the LR pair \tilde{A}, \tilde{B} and using Lemma 5.6, we see that $\{v'_i\}_{i=0}^d$ is an inverted (\tilde{A}, \tilde{B}) -basis of V^* .

(ii)–(iv) Similar to the proof of (i) above. □

Lemma 5.8. *A given LR pair A, B on V is isomorphic to the LR pair \tilde{B}, \tilde{A} on V^* .*

Proof. Let $\{\varphi_i\}_{i=1}^d$ denote the parameter sequence of A, B . By Lemma 3.14 the LR pair B, A has parameter sequence $\{\varphi_{d-i+1}\}_{i=1}^d$. Now by Lemma 5.6 the LR pair \tilde{B}, \tilde{A} has parameter sequence $\{\varphi_i\}_{i=1}^d$. The LR pairs A, B and \tilde{B}, \tilde{A} have the same parameter sequence, so they are isomorphic by Proposition 3.36. \square

6 The reflector \dagger

Throughout this section the following notation is in effect. Let V denote a vector space over \mathbb{F} with dimension $d + 1$. Let A, B denote an LR pair on V . We discuss a certain antiautomorphism \dagger of $\text{End}(V)$ called the reflector for A, B .

Proposition 6.1. *There exists a unique antiautomorphism \dagger of $\text{End}(V)$ such that $A^\dagger = B$ and $B^\dagger = A$. Moreover $(X^\dagger)^\dagger = X$ for all $X \in \text{End}(V)$.*

Proof. We first show that \dagger exists. By Lemma 5.8 there exists an isomorphism σ of LR pairs from A, B to \tilde{B}, \tilde{A} . Thus $\sigma : V \rightarrow V^*$ is an \mathbb{F} -linear bijection such that $\sigma A = \tilde{B}\sigma$ and $\sigma B = \tilde{A}\sigma$. By construction the map $\text{End}(V) \rightarrow \text{End}(V^*)$, $X \mapsto \sigma X \sigma^{-1}$ is an \mathbb{F} -algebra isomorphism that sends $A \mapsto \tilde{B}$ and $B \mapsto \tilde{A}$. Recall that the adjoint map $\text{End}(V) \rightarrow \text{End}(V^*)$, $X \mapsto \tilde{X}$ is an \mathbb{F} -algebra antiisomorphism. By these comments the composition

$$\dagger : \text{End}(V) \xrightarrow{\text{adj}} \text{End}(V^*) \xrightarrow{X \mapsto \sigma^{-1} X \sigma} \text{End}(V)$$

is an antiautomorphism of $\text{End}(V)$ such that $A^\dagger = B$ and $B^\dagger = A$. We have shown that \dagger exists. We now show that \dagger is unique. Let μ denote an antiautomorphism of $\text{End}(V)$ such that $A^\mu = B$ and $B^\mu = A$. We show that $\dagger = \mu$. The composition

$$\text{End}(V) \xrightarrow{\dagger} \text{End}(V) \xrightarrow{\mu^{-1}} \text{End}(V) \tag{22}$$

is an \mathbb{F} -algebra isomorphism that fixes each of A, B . By this and Corollary 3.11, the map (22) fixes everything in $\text{End}(V)$ and is therefore the identity map. Consequently $\dagger = \mu$. We have shown that \dagger is unique. To obtain the last assertion of the lemma, note that \dagger^{-1} is an antiautomorphism of $\text{End}(V)$ that sends $A \leftrightarrow B$. Therefore $\dagger = \dagger^{-1}$ by the uniqueness of \dagger . Consequently $(X^\dagger)^\dagger = X$ for all $X \in \text{End}(V)$. \square

Definition 6.2. By the *reflector* for A, B (or the (A, B) -*reflector*) we mean the antiautomorphism \dagger from Proposition 6.1.

Lemma 6.3. *Assume that A, B is trivial. Then the (A, B) -reflector fixes everything in $\text{End}(V)$.*

Proof. By assumption $d = 0$, so the identity I is a basis for the \mathbb{F} -vector space $\text{End}(V)$. The (A, B) -reflector is \mathbb{F} -linear and fixes I . The result follows. \square

Lemma 6.4. *Let $\{E_i\}_{i=0}^d$ denote the idempotent sequence for A, B . Then the (A, B) -reflector fixes E_i for $0 \leq i \leq d$.*

Proof. Referring to (6), for the equation on the left apply \dagger to each side and evaluate the result using the equation on the right. \square

Lemma 6.5. *The (A, B) -reflector is the same as the (B, A) -reflector.*

Proof. By Proposition 6.1 and Definition 6.2. \square

Lemma 6.6. *Let $\tilde{\dagger}$ denote the reflector for the LR pair \tilde{A}, \tilde{B} . Then the following diagram commutes:*

$$\begin{array}{ccc} \text{End}(V) & \xrightarrow{\text{adj}} & \text{End}(V^*) \\ \dagger \downarrow & & \downarrow \tilde{\dagger} \\ \text{End}(V) & \xrightarrow[\text{adj}]{} & \text{End}(V^*) \end{array}$$

Proof. Recall from Corollary 3.11 that $\text{End}(V)$ is generated by A, B . Chase A and B around the diagram, using Proposition 6.1 and Definition 6.2. The result follows. \square

7 The inverter Ψ

Throughout this section the following notation is in effect. Let V denote a vector space over \mathbb{F} with dimension $d+1$. Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$ and idempotent sequence $\{E_i\}_{i=0}^d$. We discuss a map $\Psi \in \text{End}(V)$ called the inverter for A, B . The name is motivated by Proposition 7.19 below.

Definition 7.1. Define

$$\Psi = \sum_{i=0}^d \frac{\varphi_1 \varphi_2 \cdots \varphi_i}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-i+1}} E_i. \quad (23)$$

We call Ψ the *inverter* for A, B or the (A, B) -*inverter*.

Lemma 7.2. *Assume that A, B is trivial. Then $\Psi = I$.*

Proof. In Definition 7.1 set $d = 0$ and note that $E_0 = I$. \square

Lemma 7.3. *The map Ψ is invertible, and*

$$\Psi^{-1} = \sum_{i=0}^d \frac{\varphi_d \varphi_{d-1} \cdots \varphi_{d-i+1}}{\varphi_1 \varphi_2 \cdots \varphi_i} E_i. \quad (24)$$

Proof. Use the fact that $E_i E_j = \delta_{i,j} E_i$ for $0 \leq i, j \leq d$ and $I = \sum_{i=0}^d E_i$. \square

Lemma 7.4. *Referring to the sum (23), for $0 \leq i \leq d$ the coefficients of E_i and E_{d-i} are the same; in other words*

$$\frac{\varphi_1 \varphi_2 \cdots \varphi_i}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-i+1}} = \frac{\varphi_1 \varphi_2 \cdots \varphi_{d-i}}{\varphi_d \varphi_{d-1} \cdots \varphi_{i+1}}. \quad (25)$$

Proof. Line (25) is readily checked. □

Lemma 7.5. For $0 \leq i \leq d$,

$$\Psi E_i = E_i \Psi = \frac{\varphi_1 \varphi_2 \cdots \varphi_i}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-i+1}} E_i. \quad (26)$$

Proof. Use Definition 7.1. □

Corollary 7.6. For $0 \leq i \leq d$ the following hold on $E_i V$:

$$\Psi = \frac{\varphi_1 \varphi_2 \cdots \varphi_i}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-i+1}} I, \quad \Psi^{-1} = \frac{\varphi_d \varphi_{d-1} \cdots \varphi_{d-i+1}}{\varphi_1 \varphi_2 \cdots \varphi_i} I. \quad (27)$$

Proof. Use Lemma 7.5. □

Lemma 7.7. The map Ψ fixes the (A, B) -decomposition of V and the (B, A) -decomposition of V .

Proof. The sequence $\{E_i V\}_{i=0}^d$ is the (A, B) -decomposition of V . The sequence $\{E_{d-i} V\}_{i=0}^d$ is the (B, A) -decomposition of V . By Corollary 7.6, $\Psi E_i V = E_i V$ for $0 \leq i \leq d$. The result follows. □

Lemma 7.8. The map Ψ fixes each of the following flags:

$$\{A^{d-i} V\}_{i=0}^d, \quad \{B^{d-i} V\}_{i=0}^d.$$

Proof. The flag on the left (resp. right) is induced by the (A, B) -decomposition (resp. (B, A) -decomposition) of V . The result follows in view of Lemma 7.7. □

Lemma 7.9. The map Ψ commutes with AB and BA .

Proof. By Lemma 3.16, E_i commutes with AB and BA for $0 \leq i \leq d$. The result follows in view of Definition 7.1. □

Lemma 7.10. The map Ψ is fixed by the (A, B) -reflector \dagger from Proposition 6.1 and Definition 6.2.

Proof. By Lemma 6.4 and Definition 7.1. □

Lemma 7.11. The following maps are inverse:

- (i) the inverter for the LR pair A, B ;
- (ii) the inverter for the LR pair B, A .

Proof. Use Lemmas 3.6, 3.14 along with Lemma 7.4. □

Lemma 7.12. For nonzero α, β in \mathbb{F} the following maps are the same:

- (i) the inverter for the LR pair A, B ;
- (ii) the inverter for the LR pair $\alpha A, \beta B$.

Proof. Use Lemmas 3.8, 3.39. □

Lemma 7.13. *The following maps are inverse:*

- (i) *the inverter for the LR pair \tilde{A}, \tilde{B} ;*
- (ii) *the adjoint of the inverter for the LR pair A, B .*

Proof. Use Lemmas 5.5, 5.6 along with Lemmas 7.3, 7.4. □

We turn our attention to the maps $\Psi A \Psi^{-1}$ and $\Psi^{-1} B \Psi$. We first consider how these maps act on $E_i V$ for $0 \leq i \leq d$.

Lemma 7.14. *The following (i), (ii) hold.*

- (i) $\Psi A \Psi^{-1}$ *is zero on $E_0 V$. Moreover for $1 \leq i \leq d$ and on $E_i V$,*

$$\Psi A \Psi^{-1} = \frac{\varphi_{d-i+1}}{\varphi_i} A. \quad (28)$$

- (ii) $\Psi^{-1} B \Psi$ *is zero on $E_d V$. Moreover for $0 \leq i \leq d-1$ and on $E_i V$,*

$$\Psi^{-1} B \Psi = \frac{\varphi_{d-i}}{\varphi_{i+1}} B. \quad (29)$$

Proof. The decomposition $\{E_i V\}_{i=0}^d$ is lowered by A and raised by B . The results follow from this and Corollary 7.6. □

Corollary 7.15. *The (A, B) -decomposition of V is lowered by $\Psi A \Psi^{-1}$ and raised by $\Psi^{-1} B \Psi$.*

Proof. By Lemma 7.14, and since the (A, B) -decomposition of V is equal to $\{E_i V\}_{i=0}^d$. □

Lemma 7.16. *In the table below, we give the matrices that represent $\Psi A \Psi^{-1}$ and $\Psi^{-1} B \Psi$ with respect to various bases for V .*

type of basis	matrix rep. of $\Psi A \Psi^{-1}$	matrix rep. of $\Psi^{-1} B \Psi$
(A, B)	$\begin{pmatrix} 0 & \varphi_d/\varphi_1 & & & \mathbf{0} \\ & 0 & \varphi_{d-1}/\varphi_2 & & \\ & & 0 & \ddots & \\ & & & \ddots & \varphi_1/\varphi_d \\ \mathbf{0} & & & & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & & & & \mathbf{0} \\ \varphi_d & 0 & & & \\ & \varphi_{d-1} & 0 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \varphi_1 \\ \mathbf{0} & & & & 0 \end{pmatrix}$
inv. (A, B)	$\begin{pmatrix} 0 & & & & \mathbf{0} \\ \varphi_1/\varphi_d & 0 & & & \\ & \varphi_2/\varphi_{d-1} & 0 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \varphi_d/\varphi_1 \\ \mathbf{0} & & & & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \varphi_1 & & & \mathbf{0} \\ & 0 & \varphi_2 & & \\ & & 0 & \ddots & \\ & & & \ddots & \varphi_d \\ \mathbf{0} & & & & 0 \end{pmatrix}$
(B, A)	$\begin{pmatrix} 0 & & & & \mathbf{0} \\ \varphi_1 & 0 & & & \\ & \varphi_2 & 0 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & \varphi_d & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \varphi_1/\varphi_d & & & \mathbf{0} \\ & 0 & \varphi_2/\varphi_{d-1} & & \\ & & 0 & \ddots & \\ & & & \ddots & \varphi_d/\varphi_1 \\ \mathbf{0} & & & & 0 \end{pmatrix}$
inv. (B, A)	$\begin{pmatrix} 0 & \varphi_d & & & \mathbf{0} \\ & 0 & \varphi_{d-1} & & \\ & & 0 & \ddots & \\ & & & \ddots & \varphi_1 \\ \mathbf{0} & & & & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & & & & \mathbf{0} \\ \varphi_d/\varphi_1 & 0 & & & \\ & \varphi_{d-1}/\varphi_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \varphi_1/\varphi_d \\ \mathbf{0} & & & & 0 \end{pmatrix}$

Proof. Use Lemmas 3.23, 3.26, 3.29, 3.32 along with Lemma 7.14. \square

Recall the reflector antiautomorphism \dagger from Proposition 6.1 and Definition 6.2.

Lemma 7.17. *The following (i)–(vii) hold:*

- (i) *the ordered pair $A, \Psi^{-1}B\Psi$ is an LR pair on V ;*
- (ii) *the $(A, \Psi^{-1}B\Psi)$ -decomposition of V is equal to the (A, B) -decomposition of V ;*
- (iii) *the idempotent sequence of $A, \Psi^{-1}B\Psi$ is equal to the idempotent sequence of A, B ;*
- (iv) *an $(A, \Psi^{-1}B\Psi)$ -basis of V is the same thing as an (A, B) -basis of V ;*
- (v) *the LR pair $A, \Psi^{-1}B\Psi$ has parameter sequence $\{\varphi_{d-i+1}\}_{i=1}^d$;*
- (vi) *for the LR pair $A, \Psi^{-1}B\Psi$ the reflector is the composition*

$$\text{End}(V) \xrightarrow{\dagger} \text{End}(V) \xrightarrow{X \mapsto \Psi^{-1}X\Psi} \text{End}(V);$$

(vii) for the LR pair $A, \Psi^{-1}B\Psi$ the inverter is Ψ^{-1} .

Proof. (i), (ii) The (A, B) -decomposition of V is lowered by A and raised by $\Psi^{-1}B\Psi$.

(iii) By (ii) above and Definition 3.5.

(iv) By (ii) above and Definition 3.21.

(v) Consider the matrices that represent A and $\Psi^{-1}B\Psi$ with respect to an (A, B) -basis of V . For A this matrix is given in Lemma 3.23. For $\Psi^{-1}B\Psi$ this matrix is given in Lemma 7.16.

(vi) Use Lemma 7.10.

(vii) By (iii), (v) above and line (24). □

Lemma 7.18. *The following (i)–(vii) hold:*

- (i) the ordered pair $\Psi A \Psi^{-1}, B$ is an LR pair on V ;
- (ii) the $(\Psi A \Psi^{-1}, B)$ -decomposition of V is equal to the (A, B) -decomposition of V ;
- (iii) the idempotent sequence of $\Psi A \Psi^{-1}, B$ is equal to the idempotent sequence of A, B ;
- (iv) a $(B, \Psi A \Psi^{-1})$ -basis of V is the same thing as a (B, A) -basis of V ;
- (v) the LR pair $\Psi A \Psi^{-1}, B$ has parameter sequence $\{\varphi_{d-i+1}\}_{i=1}^d$;
- (vi) for the LR pair $\Psi A \Psi^{-1}, B$ the reflector is the composition

$$\text{End}(V) \xrightarrow{\dagger} \text{End}(V) \xrightarrow{X \mapsto \Psi X \Psi^{-1}} \text{End}(V);$$

(vii) for the LR pair $\Psi A \Psi^{-1}, B$ the inverter is Ψ^{-1} .

Proof. Similar to the proof of Lemma 7.17. □

Proposition 7.19. *The following three LR pairs are mutually isomorphic:*

$$A, \Psi^{-1}B\Psi \qquad B, A \qquad \Psi A \Psi^{-1}, B. \tag{30}$$

Proof. The LR pairs (30) have the same parameter sequence by Lemmas 3.14, 7.17(v), 7.18(v). The result follows in view of Proposition 3.36. □

Lemma 7.20. *For $\sigma \in \text{End}(V)$ the following are equivalent:*

- (i) σ is an isomorphism of LR pairs from $A, \Psi^{-1}B\Psi$ to B, A ;
- (ii) σ is an isomorphism of LR pairs from B, A to $\Psi A \Psi^{-1}, B$;
- (iii) σ sends each (A, B) -basis of V to a (B, A) -basis of V .

Proof. (i) \Rightarrow (iii) The map σ sends each $(A, \Psi^{-1}B\Psi)$ -basis of V to a (B, A) -basis of V . Also by Lemma 7.17(iv), an $(A, \Psi^{-1}B\Psi)$ -basis of V is the same thing as an (A, B) -basis of V .
(iii) \Rightarrow (i) The matrix that represents A (resp. $\Psi^{-1}B\Psi$) with respect to an (A, B) -basis of V is equal to the matrix that represents B (resp. A) with respect to a (B, A) -basis of V .
(ii) \Rightarrow (iii) The map σ is an isomorphism of LR pairs from A, B to $B, \Psi A \Psi^{-1}$. So σ sends each (A, B) -basis of V to a $(B, \Psi A \Psi^{-1})$ -basis of V . Also by Lemma 7.18(iv), a $(B, \Psi A \Psi^{-1})$ -basis of V is the same thing as a (B, A) -basis of V .
(iii) \Rightarrow (ii) The matrix that represents B (resp. A) with respect to an (A, B) -basis of V is equal to the matrix that represents $\Psi A \Psi^{-1}$ (resp. B) with respect to a (B, A) -basis of V . \square

Lemma 7.21. *For $\sigma \in \text{End}(V)$ the following are equivalent:*

- (i) σ is an isomorphism of LR pairs from $A, \Psi^{-1}B\Psi$ to $\Psi A \Psi^{-1}, B$;
- (ii) there exists $0 \neq \zeta \in \mathbb{F}$ such that $\sigma = \zeta \Psi$.

Proof. (i) \Rightarrow (ii) Using Definition 3.35, we find that $\Psi^{-1}\sigma$ commutes with each of $A, \Psi^{-1}B\Psi$. Now by Lemma 3.38 (applied to the LR pair $A, \Psi^{-1}B\Psi$) there exists $0 \neq \zeta \in \mathbb{F}$ such that $\Psi^{-1}\sigma = \zeta I$. Therefore $\sigma = \zeta \Psi$.
(ii) \Rightarrow (i) It suffices to show that Ψ is an isomorphism of LR pairs from $A, \Psi^{-1}B\Psi$ to $\Psi A \Psi^{-1}, B$. Since Ψ is invertible the map $\Psi : V \rightarrow V$ is an \mathbb{F} -linear bijection. Observe that $\Psi A = (\Psi A \Psi^{-1})\Psi$ and $\Psi(\Psi^{-1}B\Psi) = B\Psi$. Now by Definition 3.35, Ψ is an isomorphism of LR pairs from $A, \Psi^{-1}B\Psi$ to $\Psi A \Psi^{-1}, B$. \square

Lemma 7.22. *The LR pairs (30) all have the same inverter.*

Proof. By Lemmas 7.11, 7.17(vii), 7.18(vii). \square

8 The outer and inner part

Throughout this section the following assumptions are in effect. We assume that $d = 2m$ is even. Let V denote a vector space over \mathbb{F} with dimension $d + 1$. Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$, idempotent sequence $\{E_i\}_{i=0}^d$, and inverter Ψ . Note that $\{E_i V\}_{i=0}^d$ is the (A, B) -decomposition of V , which is lowered by A and raised by B .

Definition 8.1. Define

$$V_{\text{out}} = \sum_{j=0}^m E_{2j} V, \quad V_{\text{in}} = \sum_{j=0}^{m-1} E_{2j+1} V.$$

Lemma 8.2. *We have*

$$V = V_{\text{out}} + V_{\text{in}} \quad (\text{direct sum}). \quad (31)$$

Moreover

$$\dim(V_{\text{out}}) = m + 1, \quad \dim(V_{\text{in}}) = m.$$

Proof. Since $\{E_i V\}_{i=0}^d$ is a decomposition of V . □

Lemma 8.3. *We have $V_{\text{out}} \neq 0$ and $V_{\text{in}} \neq V$. Moreover the following are equivalent: (i) A, B is trivial; (ii) $V_{\text{out}} = V$; (iii) $V_{\text{in}} = 0$.*

Proof. Use Example 3.4 and Lemma 8.2. □

Lemma 8.4. *The following (i)–(iii) hold:*

- (i) *for even i ($0 \leq i \leq d$), the map E_i leaves V_{out} invariant, and is zero on V_{in} ;*
- (ii) *for odd i ($0 \leq i \leq d$), the map E_i leaves V_{in} invariant, and is zero on V_{out} ;*
- (iii) *each of $V_{\text{out}}, V_{\text{in}}$ is invariant under Ψ .*

Proof. (i), (ii) Use Definition 8.1.

(iii) By (i), (ii) above and Definition 7.1. □

Lemma 8.5. *Referring to Definition 8.1,*

$$AV_{\text{out}} = V_{\text{in}}, \quad AV_{\text{in}} \subseteq V_{\text{out}}, \quad BV_{\text{out}} = V_{\text{in}}, \quad BV_{\text{in}} \subseteq V_{\text{out}}.$$

Moreover

$$A^2 V_{\text{out}} \subseteq V_{\text{out}}, \quad A^2 V_{\text{in}} \subseteq V_{\text{in}}, \quad B^2 V_{\text{out}} \subseteq V_{\text{out}}, \quad B^2 V_{\text{in}} \subseteq V_{\text{in}}.$$

Proof. By Definition 8.1 and the construction. □

Definition 8.6. Referring to Definition 8.1, the subspace V_{out} (resp. V_{in}) will be called the *outer part* (resp. *inner part*) of V with respect to A, B .

Lemma 8.7. *The outer part of V with respect to A, B coincides with the outer part of V with respect to B, A . Moreover, the inner part of V with respect to A, B coincides with the inner part of V with respect to B, A .*

Proof. By Lemma 3.6 and Definition 8.1, along with the assumption that d is even. □

Lemma 8.8. *Assume that A, B is nontrivial. Then A and B are nonzero on both V_{out} and V_{in} .*

Proof. By Definition 8.1 and the construction. □

Definition 8.9. Using the LR pair A, B we define

$$A_{\text{out}}, \quad A_{\text{in}}, \quad B_{\text{out}}, \quad B_{\text{in}} \tag{32}$$

in $\text{End}(V)$ as follows. The map A_{out} (resp. B_{out}) acts on V_{out} as A (resp. B), and on V_{in} as zero. The map A_{in} (resp. B_{in}) acts on V_{in} as A (resp. B), and on V_{out} as zero. By construction

$$A = A_{\text{out}} + A_{\text{in}}, \quad B = B_{\text{out}} + B_{\text{in}}.$$

Lemma 8.10. *Assume that A, B is nontrivial. Then*

- (i) the maps $A_{\text{out}}, A_{\text{in}}$ are linearly independent over \mathbb{F} ;
- (ii) the maps $B_{\text{out}}, B_{\text{in}}$ are linearly independent over \mathbb{F} .

Proof. (i) Suppose we are given $r, s \in \mathbb{F}$ such that $rA_{\text{out}} + sA_{\text{in}} = 0$. In this equation apply each side to V_{out} , to find $rA = 0$ on V_{out} . By Lemma 8.8 $A \neq 0$ on V_{out} . Therefore $r = 0$. One similarly shows that $s = 0$.

(ii) Similar to the proof of (i) above. □

Definition 8.11. Define

$$\Psi_{\text{out}} = \sum_{j=0}^m \frac{\varphi_1 \varphi_2 \cdots \varphi_{2j}}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-2j+1}} E_{2j}, \quad \Psi_{\text{in}} = \sum_{j=0}^{m-1} \frac{\varphi_2 \varphi_3 \cdots \varphi_{2j+1}}{\varphi_{d-1} \varphi_{d-2} \cdots \varphi_{d-2j}} E_{2j+1}.$$

Lemma 8.12. The following (i)–(iv) hold:

- (i) the subspace V_{out} is invariant under Ψ_{out} ;
- (ii) Ψ_{out} is zero on V_{in} ;
- (iii) the subspace V_{in} is invariant under Ψ_{in} ;
- (iv) Ψ_{in} is zero on V_{out} .

Proof. By Lemma 8.4(i),(ii) and Definition 8.11. □

The following two propositions are obtained by routine computation.

Proposition 8.13. The elements A^2, B^2 act on V_{out} as an LR pair. For this LR pair,

- (i) the diameter is m ;
- (ii) the parameter sequence is $\{\varphi_{2j-1} \varphi_{2j}\}_{j=1}^m$;
- (iii) the idempotent sequence is given by the actions of $\{E_{2j}\}_{j=0}^m$ on V_{out} ;
- (iv) the inverter is equal to the action of Ψ_{out} on V_{out} .

Proposition 8.14. Assume that A, B is nontrivial. Then A^2, B^2 act on V_{in} as an LR pair. For this LR pair,

- (i) the diameter is $m - 1$;
- (ii) the parameter sequence is $\{\varphi_{2j} \varphi_{2j+1}\}_{j=1}^{m-1}$;
- (iii) the idempotent sequence is given by the actions of $\{E_{2j+1}\}_{j=0}^{m-1}$ on V_{in} ;
- (iv) the inverter is equal to the action of Ψ_{in} on V_{in} .

Lemma 8.15. The maps $\Psi_{\text{out}}, \Psi_{\text{in}}$ are fixed by the (A, B) -reflector \dagger from Proposition 6.1 and Definition 6.2.

Proof. By Lemma 6.4 and Definition 8.11. □

Lemma 8.16. *Assume that A, B is nontrivial. Then*

$$\Psi = \Psi_{\text{out}} + \frac{\varphi_1}{\varphi_d} \Psi_{\text{in}}.$$

Proof. Compare Definitions 7.1, 8.11. □

Definition 8.17. We call Ψ_{out} (resp. Ψ_{in}) the *outer inverter* (resp. *inner inverter*) for the LR pair A, B .

9 The projector J

Throughout this section the following assumptions are in effect. We assume that $d = 2m$ is even. Let V denote a vector space over \mathbb{F} with dimension $d + 1$. Let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$, idempotent sequence $\{E_i\}_{i=0}^d$, and inverter Ψ . Recall the subspaces V_{out} and V_{in} from Definition 8.1.

Definition 9.1. Define $J \in \text{End}(V)$ such that $(J - I)V_{\text{out}} = 0$ and $JV_{\text{in}} = 0$. Referring to (31), the map J (resp. $I - J$) acts as the projection from V onto V_{out} (resp. V_{in}). We call J (resp. $I - J$) the *outer projector* (resp. *inner projector*) for the LR pair A, B . By the *projector* for A, B we mean the outer projector.

Lemma 9.2. *The map $J \neq 0$. If A, B is trivial then $J = I$. If A, B is nontrivial then J, I are linearly independent over \mathbb{F} .*

Proof. Use (31) and Lemma 8.3. □

Lemma 9.3. *The following (i)–(v) hold:*

- (i) $J = \sum_{j=0}^{d/2} E_{2j}$;
- (ii) $J^2 = J$;
- (iii) for even i ($0 \leq i \leq d$), $E_i J = J E_i = E_i$;
- (iv) for odd i ($0 \leq i \leq d$), $E_i J = J E_i = 0$;
- (v) $V_{\text{out}} = JV$ and $V_{\text{in}} = (I - J)V$.

Proof. (i) For the given equation the two sides agree on $E_i V$ for $0 \leq i \leq d$.

(ii)–(iv) Use (i) above and $E_r E_s = \delta_{r,s} E_r$ for $0 \leq r, s \leq d$.

(v) By Definition 9.1. □

Lemma 9.4. *For the map J (resp. $I - J$) the rank and trace are equal to $m + 1$ (resp. m).*

Proof. By Lemma 8.2, Definition 9.1, and linear algebra. □

Lemma 9.5. *The map J is fixed by the (A, B) -reflector \dagger from Proposition 6.1 and Definition 6.2.*

Proof. By Lemmas 6.4, 9.3(i). □

Lemma 9.6. *The following maps are the same:*

- (i) *the projector for the LR pair A, B ;*
- (ii) *the projector for the LR pair B, A .*

Proof. By Lemma 8.7 and Definition 9.1. □

Lemma 9.7. *For nonzero $\alpha, \beta \in \mathbb{F}$ the following maps are the same:*

- (i) *the projector for the LR pair A, B ;*
- (ii) *the projector for the LR pair $\alpha A, \beta B$.*

Proof. By Lemma 3.8 and Lemma 9.3(i). □

Lemma 9.8. *The following maps are the same:*

- (i) *the projector for the LR pair \tilde{A}, \tilde{B} ;*
- (ii) *the adjoint of the projector for A, B .*

Proof. By Lemma 5.5 and Lemma 9.3(i). □

Lemma 9.9. *Referring to Definition 8.9 the following (i)–(iii) hold:*

- (i) $A_{\text{out}} = AJ = (I - J)A$ and $B_{\text{out}} = BJ = (I - J)B$;
- (ii) $A_{\text{in}} = JA = A(I - J)$ and $B_{\text{in}} = JB = B(I - J)$;
- (iii) $A = AJ + JA$ and $B = BJ + JB$.

Proof. (i), (ii) For each given equation the two sides agree on V_{out} and V_{in} .
 (iii) By (i) above. □

Lemma 9.10. *J commutes with each of A^2, B^2, AB, BA .*

Proof. Use Lemma 9.9(iii). □

Lemma 9.11. *Referring to Definition 8.11 the following (i), (ii) hold.*

- (i) $\Psi_{\text{out}} = J\Psi = \Psi J$.
- (ii) *For A, B nontrivial,*

$$\varphi_1/\varphi_d \Psi_{\text{in}} = (I - J)\Psi = \Psi(I - J).$$

Proof. Use Definitions 7.1, 8.11, 9.1. □

Lemma 9.12. *Let V' denote a vector space over \mathbb{F} with dimension $d+1$, and let A', B' denote an LR pair on V' . Let J' denote the projector for A', B' . Let σ denote an isomorphism of LR pairs from A, B to A', B' . Then $\sigma J = J' \sigma$.*

Proof. By Lemma 3.37 and Lemma 9.3(i). □

10 Similarity and bisimilarity

In this section we describe two equivalence relations for LR pairs, called similarity and bisimilarity. Let V denote a vector space over \mathbb{F} with dimension $d + 1$.

Definition 10.1. Let A, B and A', B' denote LR pairs on V . These LR pairs will be called *associates* whenever there exist nonzero α, β in \mathbb{F} such that $A' = \alpha A$ and $B' = \beta B$. Associativity is an equivalence relation.

Lemma 10.2. Let A, B denote an LR pair on V . Let V' denote a vector space over \mathbb{F} with dimension $d + 1$, and let A', B' denote an LR pair on V' . Let $\sigma : V \rightarrow V'$ denote an \mathbb{F} -linear bijection. Then for nonzero α, β in \mathbb{F} the following (i)–(iii) are equivalent:

- (i) σ is an isomorphism of LR pairs from $\alpha A, \beta B$ to A', B' ;
- (ii) σ is an isomorphism of LR pairs from A, B to $A'/\alpha, B'/\beta$;
- (iii) $\alpha\sigma A = A'\sigma$ and $\beta\sigma B = B'\sigma$.

Proof. By Definition 3.35, assertions (i), (iii) are equivalent and assertions (ii), (iii) are equivalent. \square

Lemma 10.3. Let A, B and A', B' denote LR pairs over \mathbb{F} . Then the following are equivalent:

- (i) there exists an LR pair over \mathbb{F} that is associate to A, B and isomorphic to A', B' ;
- (ii) there exists an LR pair over \mathbb{F} that is isomorphic to A, B and associate to A', B' .

Proof. Pick nonzero α, β in \mathbb{F} . By Lemma 10.2(i),(ii) the LR pair $\alpha A, \beta B$ satisfies condition (i) in the present lemma if and only if the LR pair $A'/\alpha, B'/\beta$ satisfies condition (ii) in the present lemma. The result follows. \square

Definition 10.4. Let A, B and A', B' denote LR pairs over \mathbb{F} . These LR pairs will be called *similar* whenever they satisfy the equivalent conditions (i), (ii) in Lemma 10.3. Similarity is an equivalence relation.

Lemma 10.5. Let A, B (resp. A', B') denote an LR pair over \mathbb{F} , with parameter sequence $\{\varphi_i\}_{i=1}^d$ (resp. $\{\varphi'_i\}_{i=1}^d$). Then the following are equivalent:

- (i) the LR pairs A, B and A', B' are similar;
- (ii) the ratio φ'_i/φ_i is independent of i for $1 \leq i \leq d$.

Proof. (i) \Rightarrow (ii) By Lemma 10.3 and Definition 10.4, there exist nonzero α, β in \mathbb{F} such that $\alpha A, \beta B$ is isomorphic to A', B' . Recall from Lemma 3.39 that $\alpha A, \beta B$ has parameter sequence $\{\alpha\beta\varphi_i\}_{i=1}^d$. Now by Proposition 3.36, $\varphi'_i = \alpha\beta\varphi_i$ for $1 \leq i \leq d$. Therefore φ'_i/φ_i is independent of i for $1 \leq i \leq d$.

(ii) \Rightarrow (i) Let α denote the common value of φ'_i/φ_i for $1 \leq i \leq d$. By Lemma 3.39 and the construction, the LR pairs $\alpha A, B$ and A', B' have the same parameter sequence $\{\varphi'_i\}_{i=1}^d$. Therefore they are isomorphic by Proposition 3.36. Now the LR pairs A, B and A', B' are similar by Lemma 10.3 and Definition 10.4. \square

We now describe the bisimilarity relation. For the rest of this section, assume that $d = 2m$ is even. Until further notice let A, B denote an LR pair on V , with parameter sequence $\{\varphi_i\}_{i=1}^d$ and idempotent sequence $\{E_i\}_{i=0}^d$. Recall that $\{E_i V\}_{i=0}^d$ is the (A, B) -decomposition of V , which is lowered by A and raised by B .

Lemma 10.6. *Let V' denote a vector space over \mathbb{F} with dimension $d + 1$, and let A', B' denote an LR pair on V' . Let $\sigma : V \rightarrow V'$ denote an \mathbb{F} -linear bijection. Then the following are equivalent:*

- (i) σ is an isomorphism of LR pairs from A, B to A', B' ;
- (ii) all of

$$\sigma A_{\text{out}} = A'_{\text{out}} \sigma, \quad \sigma A_{\text{in}} = A'_{\text{in}} \sigma, \quad (33)$$

$$\sigma B_{\text{out}} = B'_{\text{out}} \sigma, \quad \sigma B_{\text{in}} = B'_{\text{in}} \sigma. \quad (34)$$

Proof. (i) \Rightarrow (ii) Use Lemma 9.9(i),(ii) and Lemma 9.12.

(ii) \Rightarrow (i) Add the two equations in (33) and use $A = A_{\text{out}} + A_{\text{in}}$, $A' = A'_{\text{out}} + A'_{\text{in}}$ to obtain $\sigma A = A' \sigma$. Similarly we obtain $\sigma B = B' \sigma$. Now by Definition 3.35 the map σ is an isomorphism of LR pairs from A, B to A', B' . \square

Lemma 10.7. *Let $\alpha_{\text{out}}, \alpha_{\text{in}}, \beta_{\text{out}}, \beta_{\text{in}}$ denote nonzero scalars in \mathbb{F} . Then the ordered pair*

$$\alpha_{\text{out}} A_{\text{out}} + \alpha_{\text{in}} A_{\text{in}}, \quad \beta_{\text{out}} B_{\text{out}} + \beta_{\text{in}} B_{\text{in}} \quad (35)$$

is an LR pair on V , with idempotent sequence $\{E_i\}_{i=0}^d$. The outer part of V with respect to (35) coincides with the outer part of V with respect to A, B . The inner part of V with respect to (35) coincides with the inner part of V with respect to A, B . The projector for the LR pair (35) coincides with the projector for A, B . The LR pair (35) has parameter sequence $\{f_i \varphi_i\}_{i=1}^d$, where

$$f_i = \begin{cases} \alpha_{\text{out}} \beta_{\text{in}} & \text{if } i \text{ is even;} \\ \alpha_{\text{in}} \beta_{\text{out}} & \text{if } i \text{ is odd} \end{cases} \quad (1 \leq i \leq d).$$

Proof. By construction. \square

Lemma 10.8. *Let A', B' denote an LR pair on V . Let $\alpha_{\text{out}}, \alpha_{\text{in}}, \beta_{\text{out}}, \beta_{\text{in}}$ denote nonzero scalars in \mathbb{F} . Then the following are equivalent:*

- (i) both

$$A' = \alpha_{\text{out}} A_{\text{out}} + \alpha_{\text{in}} A_{\text{in}}, \quad B' = \beta_{\text{out}} B_{\text{out}} + \beta_{\text{in}} B_{\text{in}};$$

- (ii) all of

$$\begin{aligned} A'_{\text{out}} &= \alpha_{\text{out}} A_{\text{out}}, & A'_{\text{in}} &= \alpha_{\text{in}} A_{\text{in}}, \\ B'_{\text{out}} &= \beta_{\text{out}} B_{\text{out}}, & B'_{\text{in}} &= \beta_{\text{in}} B_{\text{in}}. \end{aligned}$$

Proof. (i) \Rightarrow (ii) Use Definition 8.9 and Lemma 10.7.

(ii) \Rightarrow (i) By Definition 8.9 we have $A' = A'_{\text{out}} + A'_{\text{in}}$ and $B' = B'_{\text{out}} + B'_{\text{in}}$. \square

Referring to Lemmas 10.7, 10.8, we now consider the case in which $\alpha_{\text{in}} = 1$ and $\beta_{\text{in}} = 1$.

Definition 10.9. Let A', B' denote an LR pair on V . The LR pairs A, B and A', B' will be called *biassociates* whenever there exist nonzero α, β in \mathbb{F} such that

$$A' = \alpha A_{\text{out}} + A_{\text{in}}, \quad B' = \beta B_{\text{out}} + B_{\text{in}}.$$

Biassociativity is an equivalence relation.

Lemma 10.10. Let V' denote a vector space over \mathbb{F} with dimension $d + 1$, and let A', B' denote an LR pair on V' . Let $\sigma : V \rightarrow V'$ denote an \mathbb{F} -linear bijection. Then for nonzero α, β in \mathbb{F} the following (i)–(iii) are equivalent:

- (i) σ is an isomorphism of LR pairs from $\alpha A_{\text{out}} + A_{\text{in}}, \beta B_{\text{out}} + B_{\text{in}}$ to A', B' ;
- (ii) σ is an isomorphism of LR pairs from A, B to $\alpha^{-1} A'_{\text{out}} + A'_{\text{in}}, \beta^{-1} B'_{\text{out}} + B'_{\text{in}}$;
- (iii) all of

$$\begin{aligned} \alpha \sigma A_{\text{out}} &= A'_{\text{out}} \sigma, & \sigma A_{\text{in}} &= A'_{\text{in}} \sigma, \\ \beta \sigma B_{\text{out}} &= B'_{\text{out}} \sigma, & \sigma B_{\text{in}} &= B'_{\text{in}} \sigma. \end{aligned}$$

Proof. By Lemmas 10.6, 10.8 the assertions (i), (iii) are equivalent, and the assertions (ii), (iii) are equivalent. \square

Lemma 10.11. Let A, B and A', B' denote LR pairs over \mathbb{F} that have diameter d . Then the following are equivalent:

- (i) there exists an LR pair over \mathbb{F} that is biassociate to A, B and isomorphic to A', B' ;
- (ii) there exists an LR pair over \mathbb{F} that is isomorphic to A, B and biassociate to A', B' .

Proof. Pick nonzero α, β in \mathbb{F} . By Lemma 10.10(i),(ii) the LR pair $\alpha A_{\text{out}} + A_{\text{in}}, \beta B_{\text{out}} + B_{\text{in}}$ satisfies condition (i) in the present lemma if and only if the LR pair $\alpha^{-1} A'_{\text{out}} + A'_{\text{in}}, \beta^{-1} B'_{\text{out}} + B'_{\text{in}}$ satisfies condition (ii) in the present lemma. The result follows. \square

Definition 10.12. Let A, B and A', B' denote LR pairs over \mathbb{F} that have diameter d . Then A, B and A', B' will be called *bisimilar* whenever the equivalent conditions (i), (ii) hold in Lemma 10.11.

Lemma 10.13. Let A, B (resp. A', B') denote an LR pair over \mathbb{F} , with parameter sequence $\{\varphi_i\}_{i=1}^d$ (resp. $\{\varphi'_i\}_{i=1}^d$). Then the following are equivalent:

- (i) the LR pairs A, B and A', B' are bisimilar;
- (ii) the ratio φ'_i / φ_i is independent of i for i even ($1 \leq i \leq d$), and independent of i for i odd ($1 \leq i \leq d$).

Proof. (i) \Rightarrow (ii) By Lemma 10.11 and Definition 10.12, there exist nonzero α, β in \mathbb{F} such that $\alpha A_{\text{out}} + A_{\text{in}}, \beta B_{\text{out}} + B_{\text{in}}$ is isomorphic to A', B' . By Lemma 10.7 the LR pair $\alpha A_{\text{out}} + A_{\text{in}}, \beta B_{\text{out}} + B_{\text{in}}$ has parameter sequence $\{f_i \varphi_i\}_{i=1}^d$, where

$$f_i = \begin{cases} \alpha & \text{if } i \text{ is even;} \\ \beta & \text{if } i \text{ is odd} \end{cases} \quad (1 \leq i \leq d).$$

By Proposition 3.36 $\varphi'_i = f_i \varphi_i$ for $1 \leq i \leq d$. By these comments φ'_i/φ_i is independent of i for i even ($1 \leq i \leq d$), and independent of i for i odd ($1 \leq i \leq d$).

(ii) \Rightarrow (i) By assumption there exist nonzero α, β in \mathbb{F} such that

$$\varphi'_i/\varphi_i = \begin{cases} \alpha & \text{if } i \text{ is even;} \\ \beta & \text{if } i \text{ is odd} \end{cases} \quad (1 \leq i \leq d).$$

By Lemma 10.7 and the construction, the LR pairs $\alpha A_{\text{out}} + A_{\text{in}}, \beta B_{\text{out}} + B_{\text{in}}$ and A', B' have the same parameter sequence $\{\varphi'_i\}_{i=1}^d$. Therefore they are isomorphic by Proposition 3.36. Now A, B and A', B' are bisimilar in view of Lemma 10.11 and Definition 10.12. \square

11 Constrained sequences

In this section we consider a type of finite sequence, said to be constrained. We classify the constrained sequences. This classification will be used later in the paper.

Throughout this section, n denotes a nonnegative integer and $\{\rho_i\}_{i=0}^n$ denotes a sequence of scalars taken from \mathbb{F} .

Definition 11.1. The sequence $\{\rho_i\}_{i=0}^n$ is said to be *constrained* whenever

- (i) $\rho_i \rho_{n-i} = 1$ for $0 \leq i \leq n$;
- (ii) there exist $a, b, c \in \mathbb{F}$ that are not all zero and $a\rho_{i-1} + b\rho_i + c\rho_{i+1} = 0$ for $1 \leq i \leq n-1$.

Shortly we will classify the constrained sequences. We will use an inductive argument based on the following observation.

Lemma 11.2. Assume that $n \geq 2$ and the sequence $\{\rho_i\}_{i=0}^n$ is constrained. Then the sequence $\rho_1, \rho_2, \dots, \rho_{n-1}$ is constrained.

The sequence $\{\rho_i\}_{i=0}^n$ is called *geometric* whenever $\rho_i \neq 0$ for $0 \leq i \leq n$ and ρ_i/ρ_{i-1} is independent of i for $1 \leq i \leq n$. The following are equivalent: (i) $\{\rho_i\}_{i=0}^n$ is geometric; (ii) there exist nonzero $r, \xi \in \mathbb{F}$ such that $\rho_i = \xi r^i$ for $0 \leq i \leq n$. In this case $\xi = \rho_0$ and $r = \rho_i/\rho_{i-1}$ for $1 \leq i \leq n$.

We now classify the constrained sequences. The case of n even and n odd will be treated separately.

Proposition 11.3. Assume that n is even. Then for the sequence $\{\rho_i\}_{i=0}^n$ the following (i)–(iii) are equivalent:

- (i) $\{\rho_i\}_{i=0}^n$ is constrained;
- (ii) $\{\rho_i\}_{i=0}^n$ is geometric and $\rho_{n/2} \in \{1, -1\}$;
- (iii) there exist $0 \neq r \in \mathbb{F}$ and $\varepsilon \in \{1, -1\}$ such that $\rho_i = \varepsilon r^{i-n/2}$ for $0 \leq i \leq n$.

Assume that (i)–(iii) hold. Then $r = \rho_i/\rho_{i-1}$ for $1 \leq i \leq n$, and $\varepsilon = \rho_{n/2}$.

Proof. (i) \Rightarrow (iii) Our proof is by induction on n . First assume that $n = 0$. Then $\rho_0^2 = 1$. Condition (iii) holds with $\varepsilon = \rho_0$ and arbitrary $0 \neq r \in \mathbb{F}$. Next assume that $n = 2$. Then $\rho_0\rho_2 = 1$ and $\rho_1^2 = 1$. Condition (iii) holds with $\varepsilon = \rho_1$ and $r = \rho_1/\rho_0$. Next assume that $n \geq 4$. By Definition 11.1(ii), there exist $a, b, c \in \mathbb{F}$ that are not all zero and $a\rho_{i-1} + b\rho_i + c\rho_{i+1} = 0$ for $1 \leq i \leq n-1$. Define $m = n-2$ and $\rho'_i = \rho_{i+1}$ for $0 \leq i \leq m$. By construction $\rho'_i\rho'_{m-i} = 1$ for $0 \leq i \leq m$. Moreover $a\rho'_{i-1} + b\rho'_i + c\rho'_{i+1} = 0$ for $1 \leq i \leq m-1$. By induction there exist $0 \neq r \in \mathbb{F}$ and $\varepsilon \in \{1, -1\}$ such that $\rho'_i = \varepsilon r^{i-m/2}$ for $0 \leq i \leq m$. Now

$$\rho_i = \varepsilon r^{i-n/2} \quad (1 \leq i \leq n-1). \quad (36)$$

We show that $\rho_0 = \varepsilon r^{-n/2}$ and $\rho_n = \varepsilon r^{n/2}$. Since $\rho_0\rho_n = 1$, it suffices to show that $\rho_0 = \varepsilon r^{-n/2}$. We claim that

$$\det \begin{pmatrix} \rho_0 & \rho_1 & \rho_{n-2} \\ \rho_1 & \rho_2 & \rho_{n-1} \\ \rho_2 & \rho_3 & \rho_n \end{pmatrix} = \frac{-r^2(\rho_0 - \varepsilon r^{-n/2})^2}{\rho_0}. \quad (37)$$

To verify (37), evaluate the determinant using (36) and $\rho_0\rho_n = 1$, and simplify the result. The claim is proven. For the matrix in (37), $a(\text{top row}) + b(\text{middle row}) + c(\text{bottom row}) = 0$. The matrix is singular, so its determinant is zero. Therefore $\rho_0 = \varepsilon r^{-n/2}$ as desired.

(iii) \Rightarrow (i) By construction $\rho_i\rho_{n-i} = 1$ for $0 \leq i \leq n$. Define $a = r$, $b = -1$, $c = 0$. Then $a\rho_{i-1} + b\rho_i + c\rho_{i+1} = 0$ for $1 \leq i \leq n-1$.

(ii) \Leftrightarrow (iii) Routine. \square

Proposition 11.4. Assume that n is odd. Then for the sequence $\{\rho_i\}_{i=0}^n$ the following (i)–(iii) are equivalent:

- (i) $\{\rho_i\}_{i=0}^n$ is constrained;
- (ii) the sequences $\rho_0, \rho_2, \dots, \rho_{n-1}$ and $\rho_n^{-1}, \rho_{n-2}^{-1}, \dots, \rho_1^{-1}$ are equal and geometric;
- (iii) there exist nonzero $s, \xi \in \mathbb{F}$ such that

$$\rho_i = \begin{cases} \xi s^{i/2} & \text{if } i \text{ is even;} \\ \xi^{-1} s^{(i-n)/2} & \text{if } i \text{ is odd} \end{cases} \quad (0 \leq i \leq n). \quad (38)$$

Assume that (i)–(iii) hold. Then $s = \rho_i/\rho_{i-2}$ for $2 \leq i \leq n$ and $\xi = \rho_0$.

Proof. (i) \Rightarrow (iii) Our proof is by induction on n . First assume that $n = 1$. Then (iii) holds with $\xi = \rho_0$ and arbitrary $0 \neq s \in \mathbb{F}$. Next assume that $n = 3$. Then (iii) holds with $\xi = \rho_0$ and $s = \rho_2/\rho_0$. Next assume that $n \geq 5$. By Definition 11.1(ii), there exist $a, b, c \in \mathbb{F}$ that are not all zero and $a\rho_{i-1} + b\rho_i + c\rho_{i+1} = 0$ for $1 \leq i \leq n-1$. Define $m = n-2$ and $\rho'_i = \rho_{i+1}$ for $0 \leq i \leq m$. By construction $\rho'_i \rho'_{m-i} = 1$ for $0 \leq i \leq m$. Moreover $a\rho'_{i-1} + b\rho'_i + c\rho'_{i+1} = 0$ for $1 \leq i \leq m-1$. By induction there exist nonzero $s, x \in \mathbb{F}$ such that

$$\rho'_i = \begin{cases} xs^{i/2} & \text{if } i \text{ is even;} \\ x^{-1}s^{(i-m)/2} & \text{if } i \text{ is odd} \end{cases} \quad (0 \leq i \leq m).$$

Define $\xi = x^{-1}s^{-(1+m)/2}$. Then

$$\rho_i = \begin{cases} \xi s^{i/2} & \text{if } i \text{ is even;} \\ \xi^{-1}s^{(i-n)/2} & \text{if } i \text{ is odd} \end{cases} \quad (1 \leq i \leq n-1). \quad (39)$$

We show that $\rho_0 = \xi$ and $\rho_n = \xi^{-1}$. Since $\rho_0\rho_n = 1$, it suffices to show that $\rho_0 = \xi$. We claim that

$$\det \begin{pmatrix} \rho_0 & \rho_1 & \rho_2 \\ \rho_1 & \rho_2 & \rho_3 \\ \rho_2 & \rho_3 & \rho_4 \end{pmatrix} + \xi^2 s^2 \det \begin{pmatrix} \rho_0 & \rho_1 & \rho_{n-2} \\ \rho_1 & \rho_2 & \rho_{n-1} \\ \rho_2 & \rho_3 & \rho_n \end{pmatrix} = \frac{-(\rho_0 - \xi)^2}{s^{n-3}\xi^2\rho_0}. \quad (40)$$

To verify (40), evaluate the two determinants using (39) and $\rho_0\rho_n = 1$, and simplify the result. The claim is proven. For each matrix in (40), $a(\text{top row}) + b(\text{middle row}) + c(\text{bottom row}) = 0$. Each matrix is singular, so its determinant is zero. Therefore $\rho_0 = \xi$.

(iii) \Rightarrow (i) By construction $\rho_i\rho_{n-i} = 1$ for $0 \leq i \leq n$. Define $a = s$, $b = 0$, $c = -1$. Then $a\rho_{i-1} + b\rho_i + c\rho_{i+1} = 0$ for $1 \leq i \leq n-1$.

(ii) \Leftrightarrow (iii) Routine. \square

Assume for the moment that n is odd, and refer to Proposition 11.4. It could happen that $\{\rho_i\}_{i=0}^n$ is geometric, and the equivalent conditions (i)–(iii) hold. We now investigate this case.

Lemma 11.5. *Assume that n is odd. Then for the sequence $\{\rho_i\}_{i=0}^n$ the following (i)–(iv) are equivalent:*

- (i) $\{\rho_i\}_{i=0}^n$ is geometric and constrained;
- (ii) $\{\rho_i\}_{i=0}^n$ is geometric and $\rho_{(n-1)/2}, \rho_{(n+1)/2}$ are inverses;
- (iii) there exists $0 \neq t \in \mathbb{F}$ such that $\rho_i = t^{2i-n}$ for $0 \leq i \leq n$;
- (iv) there exist $s, \xi \in \mathbb{F}$ that satisfy $\xi^4 s^n = 1$ and Proposition 11.4(iii).

Assume that (i)–(iv) hold. Then $\xi = \rho_0 = t^{-n}$ and

$$s = t^4, \quad \rho_{(n-1)/2} = t^{-1}, \quad \rho_{(n+1)/2} = t, \quad t^2 = \rho_i/\rho_{i-1} \quad (1 \leq i \leq n). \quad (41)$$

Proof. (i) \Rightarrow (ii) By Definition 11.1(i).

(ii) \Rightarrow (iii) By assumption there exists $0 \neq t \in \mathbb{F}$ such that $\rho_{(n-1)/2} = t^{-1}$ and $\rho_{(n+1)/2} = t$. Since $\{\rho_i\}_{i=0}^n$ is geometric, the ratio ρ_i/ρ_{i-1} is independent of i for $1 \leq i \leq n$. Taking $i = (n+1)/2$ we find that this ratio is t^2 . By these comments $\rho_i = t^{2i-n}$ for $0 \leq i \leq n$.

(iii) \Rightarrow (iv) The values $s = t^4$, $\xi = t^{-n}$ meet the requirement.

(iv) \Rightarrow (i) Evaluating (38) using $\xi^4 s^n = 1$ we find that $\{\rho_i\}_{i=0}^n$ is geometric. The sequence $\{\rho_i\}_{i=0}^n$ is constrained by Proposition 11.4(i),(iii).

Assume that (i)–(iv) hold. Then the equations $\xi = \rho_0 = t^{-n}$ and (41) are readily checked. \square

Lemma 11.6. *Assume that $\{\rho_i\}_{i=0}^n$ is constrained but not geometric. Then n is odd and at least 3.*

Proof. The integer n is odd by Proposition 11.3(i),(ii). If $n = 1$ then $\rho_0 \rho_1 = 1$, and the sequence ρ_0, ρ_1 is geometric, for a contradiction. Therefore $n \geq 3$. \square

We have some comments about the scalars a, b, c from Definition 11.1(ii).

Definition 11.7. Assume that the sequence $\{\rho_i\}_{i=0}^n$ is constrained. By a *linear constraint* for this sequence we mean a vector $(a, b, c) \in \mathbb{F}^3$ such that $a\rho_{i-1} + b\rho_i + c\rho_{i+1} = 0$ for $1 \leq i \leq n-1$.

Let λ denote an indeterminate, and let $\mathbb{F}[\lambda]$ denote the \mathbb{F} -algebra consisting of the polynomials in λ that have all coefficients in \mathbb{F} . Let (a, b, c) denote a vector in \mathbb{F}^3 . Define a polynomial $\psi \in \mathbb{F}[\lambda]$ by

$$\psi = a + b\lambda + c\lambda^2.$$

Proposition 11.8. *Assume that $n \geq 2$ and $\{\rho_i\}_{i=0}^n$ is constrained. Then for the above vector (a, b, c) the following (i)–(iii) hold.*

- (i) *Assume that n is even. Then (a, b, c) is a linear constraint for $\{\rho_i\}_{i=0}^n$ if and only if $\psi(r) = 0$, where r is from Proposition 11.3.*
- (ii) *Assume that n is odd and $\{\rho_i\}_{i=0}^n$ is not geometric. Then (a, b, c) is a linear constraint for $\{\rho_i\}_{i=0}^n$ if and only if $\psi = c(\lambda^2 - s)$, where s is from Proposition 11.4.*
- (iii) *Assume that n is odd and $\{\rho_i\}_{i=0}^n$ is geometric. Then (a, b, c) is a linear constraint for $\{\rho_i\}_{i=0}^n$ if and only if $\psi(t^2) = 0$, where t is from Lemma 11.5.*

Proof. (i) By Proposition 11.3 we have $\rho_i = \varepsilon r^{i-n/2}$ for $0 \leq i \leq n$, where $r = \rho_i/\rho_{i-1}$ for $1 \leq i \leq n$ and $\varepsilon = \rho_{n/2}$. For $1 \leq i \leq n-1$, $a\rho_{i-1} + b\rho_i + c\rho_{i+1} = 0$ if and only if $a + br + cr^2 = 0$ if and only if $\psi(r) = 0$. So (a, b, c) is a linear constraint for $\{\rho_i\}_{i=0}^n$ if and only if $a\rho_{i-1} + b\rho_i + c\rho_{i+1} = 0$ for $1 \leq i \leq n-1$ if and only if $\psi(r) = 0$.

(ii) The sequence $\{\rho_i\}_{i=0}^n$ is constrained, so by Proposition 11.4 it has the form (38), where $s = \rho_i/\rho_{i-2}$ for $2 \leq i \leq n$ and $\xi = \rho_0$. By assumption $\{\rho_i\}_{i=0}^n$ is not geometric, so $\xi^4 s^n \neq 1$ in view of Lemma 11.5(i),(iv). Define $k = \xi^2 s^{(n+1)/2}$ and note that $k^2 = \xi^4 s^{n+1}$. Therefore $k^2 \neq s$ so $k \neq k^{-1}s$. Pick an integer i ($1 \leq i \leq n-1$). First assume that i is even. By (38), $a\rho_{i-1} + b\rho_i + c\rho_{i+1} = 0$ if and only if $a + bk + cs = 0$. Next assume that i is odd. By (38),

$a\rho_{i-1} + b\rho_i + c\rho_{i+1} = 0$ if and only if $a + bk^{-1}s + cs = 0$. We now argue that (a, b, c) is a linear constraint for $\{\rho_i\}_{i=0}^n$ if and only if $a\rho_{i-1} + b\rho_i + c\rho_{i+1} = 0$ for $1 \leq i \leq n-1$ if and only if both $a + bk + cs = 0$, $a + bk^{-1}s + cs = 0$ if and only if $b = 0 = a + cs$ if and only if $\psi = c(\lambda^2 - s)$.

(iii) Similar to the proof of (i) above. \square

Definition 11.9. Assume that the sequence $\{\rho_i\}_{i=0}^n$ is constrained. Let LC denote the set of linear constraints for $\{\rho_i\}_{i=0}^n$. By Definition 11.7, LC is a subspace of the \mathbb{F} -vector space \mathbb{F}^3 . By Definition 11.1(ii), LC is nonzero. We call LC the *linear constraint space* for $\{\rho_i\}_{i=0}^n$.

Proposition 11.10. Assume that $n \geq 2$ and $\{\rho_i\}_{i=0}^n$ is constrained. Let LC denote the corresponding linear constraint space.

- (i) Assume that n is even. Then LC has dimension 2. The vectors $(r, -1, 0)$ and $(r^2, 0, -1)$ form a basis of LC, where r is from Proposition 11.3.
- (ii) Assume that n is odd and $\{\rho_i\}_{i=0}^n$ is not geometric. Then LC has dimension 1. The vector $(s, 0, -1)$ forms a basis of LC, where s is from Proposition 11.4.
- (iii) Assume that n is odd and $\{\rho_i\}_{i=0}^n$ is geometric. Then LC has dimension 2. The vectors $(t^2, -1, 0)$ and $(t^4, 0, -1)$ form a basis of LC, where t is from Lemma 11.5.

Proof. This is a reformulation of Proposition 11.8. \square

12 Toeplitz matrices and nilpotent linear transformations

We will be discussing an upper triangular matrix of a certain type, said to be Toeplitz.

Definition 12.1. (See [7, Section 8.12].) Let $\{\alpha_i\}_{i=0}^d$ denote scalars in \mathbb{F} . Let T denote an upper triangular matrix in $\text{Mat}_{d+1}(\mathbb{F})$. Then T is said to *Toeplitz, with parameters* $\{\alpha_i\}_{i=0}^d$ whenever T has (i, j) -entry α_{j-i} for $0 \leq i \leq j \leq d$. In this case

$$T = \begin{pmatrix} \alpha_0 & \alpha_1 & \cdot & \cdot & \cdot & \alpha_d \\ & \alpha_0 & \alpha_1 & \cdot & \cdot & \cdot \\ & & \alpha_0 & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \alpha_1 \\ \mathbf{0} & & & & & \alpha_0 \end{pmatrix}.$$

We have some comments.

Note 12.2. The matrix τ from Definition 3.49 is Toeplitz, with parameters

$$\alpha_i = \begin{cases} 1 & \text{if } i = 1; \\ 0 & \text{if } i \neq 1 \end{cases} \quad (0 \leq i \leq d).$$

Lemma 12.3. *Let T denote an upper triangular matrix in $\text{Mat}_{d+1}(\mathbb{F})$. Then T is Toeplitz if and only if T commutes with τ . In this case $T = \sum_{i=0}^d \alpha_i \tau^i$, where $\{\alpha_i\}_{i=0}^d$ are the parameters for T .*

Proof. Use Lemma 3.51. □

Lemma 12.4. *Let T denote an upper triangular Toeplitz matrix in $\text{Mat}_{d+1}(\mathbb{F})$. Then $T^t = \mathbf{Z}T\mathbf{Z}$.*

Proof. Expand $\mathbf{Z}T\mathbf{Z}$ by matrix multiplication. □

Referring to Definition 12.1, assume that T is Toeplitz with parameters $\{\alpha_i\}_{i=0}^d$. Note that T is invertible if and only if $\alpha_0 \neq 0$. Assume this is the case. Then T^{-1} is upper triangular and Toeplitz:

$$T^{-1} = \begin{pmatrix} \beta_0 & \beta_1 & \cdot & \cdot & \cdot & \beta_d \\ & \beta_0 & \beta_1 & \cdot & \cdot & \cdot \\ & & \beta_0 & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \beta_1 \\ \mathbf{0} & & & & & \beta_0 \end{pmatrix},$$

with parameters $\{\beta_i\}_{i=0}^d$ that are obtained from $\{\alpha_i\}_{i=0}^d$ by recursively solving $\alpha_0\beta_0 = 1$ and

$$\alpha_0\beta_j + \alpha_1\beta_{j-1} + \cdots + \alpha_j\beta_0 = 0 \quad (1 \leq j \leq d). \quad (42)$$

We have

$$\begin{aligned} \beta_0 &= \alpha_0^{-1}, \\ \beta_1 &= -\alpha_1\alpha_0^{-2} && (\text{if } d \geq 1), \\ \beta_2 &= \frac{\alpha_1^2 - \alpha_0\alpha_2}{\alpha_0^3} && (\text{if } d \geq 2), \\ \beta_3 &= \frac{2\alpha_0\alpha_1\alpha_2 - \alpha_1^3 - \alpha_0^2\alpha_3}{\alpha_0^4} && (\text{if } d \geq 3), \\ \beta_4 &= \frac{\alpha_1^4 + 2\alpha_0^2\alpha_1\alpha_3 + \alpha_0^2\alpha_2^2 - 3\alpha_0\alpha_1^2\alpha_2 - \alpha_0^3\alpha_4}{\alpha_0^5} && (\text{if } d \geq 4). \end{aligned}$$

For $\alpha_0 = 1$ this becomes

$$\begin{aligned} \beta_0 &= 1, \\ \beta_1 &= -\alpha_1, && (\text{if } d \geq 1), \\ \beta_2 &= \alpha_1^2 - \alpha_2 && (\text{if } d \geq 2), \\ \beta_3 &= 2\alpha_1\alpha_2 - \alpha_1^3 - \alpha_3 && (\text{if } d \geq 3), \\ \beta_4 &= \alpha_1^4 + 2\alpha_1\alpha_3 + \alpha_2^2 - 3\alpha_1^2\alpha_2 - \alpha_4 && (\text{if } d \geq 4). \end{aligned}$$

Recall our vector space V over \mathbb{F} with dimension $d+1$. Recall the Nil elements in $\text{End}(V)$ from Definition 2.4. We now give a variation on Lemma 2.5(i),(ii) in terms of vectors.

Lemma 12.5. For $A \in \text{End}(V)$ the following are equivalent:

- (i) A is Nil;
- (ii) there exists a basis $\{v_i\}_{i=0}^d$ of V such that $Av_0 = 0$ and $Av_i = v_{i-1}$ for $1 \leq i \leq d$.

Proof. By Lemma 2.5(i),(ii). \square

Lemma 12.6. Assume $A \in \text{End}(V)$ is Nil. For subspaces $\{V_i\}_{i=0}^d$ of V the following are equivalent:

- (i) $\{V_i\}_{i=0}^d$ is a decomposition of V that is lowered by A ;
- (ii) the sum $V = AV + V_d$ is direct, and $V_i = A^{d-i}V_d$ for $0 \leq i \leq d$.

Proof. (i) \Rightarrow (ii) The sum $V = AV + V_d$ is direct since $AV = V_0 + \dots + V_{d-1}$. The remaining assertion is clear.

(ii) \Rightarrow (i) Define $U_i = A^{d-i}V$ for $0 \leq i \leq d$. The sequence $\{U_i\}_{i=0}^d$ is a flag on V . By construction, the sum $U_i = U_{i-1} + V_i$ is direct for $1 \leq i \leq d$. Therefore $\{V_i\}_{i=0}^d$ is a decomposition of V . By construction $AV_i = V_{i-1}$ for $1 \leq i \leq d$. Also $AV_0 = 0$ since A is Nil. By these comments $\{V_i\}_{i=0}^d$ is lowered by A . \square

Lemma 12.7. Assume $A \in \text{End}(V)$ is Nil. For vectors $\{v_i\}_{i=0}^d$ in V the following are equivalent:

- (i) $\{v_i\}_{i=0}^d$ is a basis of V such that $Av_0 = 0$ and $Av_i = v_{i-1}$ for $1 \leq i \leq d$;
- (ii) $v_d \notin AV$ and $v_i = A^{d-i}v_d$ for $0 \leq i \leq d$.

Proof. This is a reformulation of Lemma 12.6. \square

Assume $A \in \text{End}(V)$ is Nil. By Lemma 12.7, for $v \in V \setminus AV$ the sequences $\{A^i v\}_{i=0}^d$ and $\{A^{d-i} v\}_{i=0}^d$ are bases for V . Relative to these bases the matrices representing A are, respectively,

$$A : \begin{pmatrix} 0 & & & \mathbf{0} \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \\ \mathbf{0} & & & 1 & 0 \end{pmatrix}, \quad A : \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & 0 & 1 & \\ & & 0 & \ddots \\ & & & \ddots & 1 \\ \mathbf{0} & & & & 0 \end{pmatrix}.$$

For $u, v \in V \setminus AV$, we now compute the transition matrix from the basis $\{A^{d-i}u\}_{i=0}^d$ to the basis $\{A^{d-i}v\}_{i=0}^d$. There exist scalars $\{\alpha_i\}_{i=0}^d$ in \mathbb{F} such that $\alpha_0 \neq 0$ and $v = \sum_{i=0}^d \alpha_i A^i u$. In this equation, for $0 \leq j \leq d$ apply A^j to each side and adjust the result to obtain $A^j v = \sum_{i=j}^d \alpha_{i-j} A^i u$. This yields

$$A^{d-j} v = \sum_{i=0}^j \alpha_{j-i} A^{d-i} u \quad 0 \leq j \leq d. \quad (43)$$

By (43), the transition matrix from the basis $\{A^{d-i}u\}_{i=0}^d$ to the basis $\{A^{d-i}v\}_{i=0}^d$ is upper triangular, and Toeplitz with parameters $\{\alpha_i\}_{i=0}^d$. Define $\Phi = \sum_{i=0}^d \alpha_i A^i$. By construction $A\Phi = \Phi A$ and $\Phi u = v$. Therefore Φ sends $A^{d-i}u \mapsto A^{d-i}v$ for $0 \leq i \leq d$.

Proposition 12.8. Let $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$ denote bases for V . Then the following are equivalent:

- (i) there exists $A \in \text{End}(V)$ such that $Au_i = u_{i-1}$ ($1 \leq i \leq d$), $Au_0 = 0$, $Av_i = v_{i-1}$ ($1 \leq i \leq d$), $Av_0 = 0$;
- (ii) the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$ is upper triangular and Toeplitz.

Proof. (i) \Rightarrow (ii) The map A is Nil by Lemma 12.5. By Lemma 12.7, there exist $u, v \in V \setminus AV$ such that $u_i = A^{d-i}u$ and $v_i = A^{d-i}v$ for $0 \leq i \leq d$. Now (ii) follows from the sentence below (43).

(ii) \Rightarrow (i) Define $A \in \text{End}(V)$ such that $Au_0 = 0$ and $Au_i = u_{i-1}$ for $1 \leq i \leq d$. For the transition matrix in question let $\{\alpha_i\}_{i=0}^d$ denote the corresponding parameters, and define $\Phi = \sum_{i=0}^d \alpha_i A^i$. Then $A\Phi = \Phi A$. By construction $\Phi u_i = v_i$ for $0 \leq i \leq d$. Therefore $Av_0 = 0$ and $Av_i = v_{i-1}$ for $1 \leq i \leq d$. \square

Lemma 12.9. Assume that the two equivalent conditions in Proposition 12.8 hold, and let $\{\alpha_i\}_{i=0}^d$ denote the parameters for the Toeplitz matrix mentioned in the second condition. Fix $0 \neq r \in \mathbb{F}$.

- (i) If we replace u_i by $u'_i = ru_i$ for $0 \leq i \leq d$, then the equivalent conditions in Proposition 12.8 still hold, with $A' = A$ and $\alpha'_i = r^{-1}\alpha_i$ for $0 \leq i \leq d$.
- (ii) If we replace u_i and v_i by $u'_i = r^i u_i$ and $v'_i = r^i v_i$ for $0 \leq i \leq d$, then the equivalent conditions in Proposition 12.8 still hold, with $A' = r^{-1}A$ and $\alpha'_i = r^i \alpha_i$ for $0 \leq i \leq d$.

Proof. By linear algebra. \square

13 LR triples

We now turn our attention to LR triples. Throughout this section, V denotes a vector space over \mathbb{F} with dimension $d+1$.

Definition 13.1. An *LR triple* on V is a sequence A, B, C of elements in $\text{End}(V)$ such that any two of A, B, C form an LR pair on V . This LR triple is said to be *over* \mathbb{F} . We call V the *underlying vector space*. We call d the *diameter*.

Definition 13.2. Let A, B, C denote an LR triple on V . Let V' denote a vector space over \mathbb{F} with dimension $d+1$, and let A', B', C' denote an LR triple on V' . By an *isomorphism of LR triples* from A, B, C to A', B', C' we mean an \mathbb{F} -linear bijection $\sigma : V \rightarrow V'$ such that

$$\sigma A = A' \sigma, \quad \sigma B = B' \sigma, \quad \sigma C = C' \sigma.$$

The LR triples A, B, C and A', B', C' are called *isomorphic* whenever there exists an isomorphism of LR triples from A, B, C to A', B', C' .

Example 13.3. Assume $d = 0$. A sequence of elements A, B, C in $\text{End}(V)$ form an LR triple if and only if each of A, B, C is zero. This LR triple will be called *trivial*.

We will use the following notational convention.

Definition 13.4. Let A, B, C denote an LR triple. For any object f that we associate with this LR triple, then f' (resp. f'') will denote the corresponding object for the LR triple B, C, A (resp. C, A, B).

Definition 13.5. Let A, B, C denote an LR triple on V . By Definition 13.1, the pair A, B (resp. B, C) (resp. C, A) is an LR pair on V . Following the notational convention in Definition 13.4, for these LR pairs the parameter sequence is denoted as follows:

LR pair	parameter sequence
A, B	$\{\varphi_i\}_{i=1}^d$
B, C	$\{\varphi'_i\}_{i=1}^d$
C, A	$\{\varphi''_i\}_{i=1}^d$

We call the sequence

$$(\{\varphi_i\}_{i=1}^d; \{\varphi'_i\}_{i=1}^d; \{\varphi''_i\}_{i=1}^d) \quad (44)$$

the *parameter array* of the LR triple A, B, C .

Note 13.6. As we will see, not every LR triple is determined up to isomorphism by its parameter array.

Lemma 13.7. Let A, B, C denote an LR triple on V , with parameter array (44). Let α, β, γ denote nonzero scalars in \mathbb{F} . Then the triple $\alpha A, \beta B, \gamma C$ is an LR triple on V , with parameter array

$$(\{\alpha\beta\varphi_i\}_{i=1}^d; \{\beta\gamma\varphi'_i\}_{i=1}^d; \{\gamma\alpha\varphi''_i\}_{i=1}^d).$$

Proof. Use Lemma 3.39 and Definition 13.5. □

Lemma 13.8. Let A, B, C denote a nontrivial LR triple over \mathbb{F} . Let α, β, γ denote nonzero scalars in \mathbb{F} . Then the following are equivalent:

- (i) the LR triples A, B, C and $\alpha A, \beta B, \gamma C$ have the same parameter array;
- (ii) $\alpha\beta = \beta\gamma = \gamma\alpha = 1$;
- (iii) $\alpha = \beta = \gamma \in \{1, -1\}$.

Proof. Use Lemma 13.7. □

Lemma 13.9. Let A, B, C denote an LR triple on V , with parameter array (44). Then each permutation of A, B, C is an LR triple on V . The corresponding parameter array is given in the table below:

LR triple	parameter array
A, B, C	$(\{\varphi_i\}_{i=1}^d; \{\varphi'_i\}_{i=1}^d; \{\varphi''_i\}_{i=1}^d)$
B, C, A	$(\{\varphi'_i\}_{i=1}^d; \{\varphi''_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d)$
C, A, B	$(\{\varphi''_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d; \{\varphi'_i\}_{i=1}^d)$
C, B, A	$(\{\varphi'_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d; \{\varphi''_{d-i+1}\}_{i=1}^d)$
A, C, B	$(\{\varphi''_{d-i+1}\}_{i=1}^d; \{\varphi'_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d)$
B, A, C	$(\{\varphi_{d-i+1}\}_{i=1}^d; \{\varphi''_{d-i+1}\}_{i=1}^d; \{\varphi'_{d-i+1}\}_{i=1}^d)$

Proof. By Lemma 3.14 and Definition 13.5. \square

Definition 13.10. Let A, B, C and A', B', C' denote LR triples on V . These LR triples will be called *associates* whenever there exist nonzero α, β, γ in \mathbb{F} such that

$$A' = \alpha A, \quad B' = \beta B, \quad C' = \gamma C.$$

Associativity is an equivalence relation.

Lemma 13.11. Let A, B, C and A', B', C' denote LR triples over \mathbb{F} . Then the following are equivalent:

- (i) there exists an LR triple over \mathbb{F} that is associate to A, B, C and isomorphic to A', B', C' ;
- (ii) there exists an LR triple over \mathbb{F} that is isomorphic to A, B, C and associate to A', B', C' .

Proof. Similar to the proof of Lemma 10.3. \square

Definition 13.12. Let A, B, C and A', B', C' denote LR triples over \mathbb{F} . These LR triples will be called *similar* whenever they satisfy the equivalent conditions (i), (ii) in Lemma 13.11. Similarity is an equivalence relation.

Lemma 13.13. Let A, B, C denote an LR triple on V , with parameter array (44). In each row of the table below, we display an LR triple on V^* along with its parameter array.

LR triple	parameter array
$\tilde{A}, \tilde{B}, \tilde{C}$	$(\{\varphi_{d-i+1}\}_{i=1}^d; \{\varphi'_{d-i+1}\}_{i=1}^d; \{\varphi''_{d-i+1}\}_{i=1}^d)$
$\tilde{B}, \tilde{C}, \tilde{A}$	$(\{\varphi'_{d-i+1}\}_{i=1}^d; \{\varphi''_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d)$
$\tilde{C}, \tilde{A}, \tilde{B}$	$(\{\varphi''_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d; \{\varphi'_{d-i+1}\}_{i=1}^d)$
$\tilde{C}, \tilde{B}, \tilde{A}$	$(\{\varphi'_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d; \{\varphi''_i\}_{i=1}^d)$
$\tilde{A}, \tilde{C}, \tilde{B}$	$(\{\varphi''_i\}_{i=1}^d; \{\varphi'_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d)$
$\tilde{B}, \tilde{A}, \tilde{C}$	$(\{\varphi_i\}_{i=1}^d; \{\varphi''_i\}_{i=1}^d; \{\varphi'_i\}_{i=1}^d)$

Proof. The given triples are LR triples by Lemma 5.4(i) and Definition 13.1. To compute their parameter array use Lemmas 5.6, 13.9. \square

Definition 13.14. Let A, B, C denote an LR triple on V . By a *relative* of A, B, C we mean an LR triple from the table in Lemma 13.9 or Lemma 13.13. A relative of A, B, C is said to have *positive orientation* (resp. *negative orientation*) with respect to A, B, C whenever it is in the top half (resp. bottom half) of the table of Lemma 13.9 or the bottom half (resp. top half) of the table in Lemma 13.13. We call such a relative a *p-relative* (resp. *n-relative*) of A, B, C . Note that an n-relative of A, B, C is the same thing as a p-relative of C, B, A .

Let A, B, C denote an LR triple on V . By Lemma 3.3, each of A, B, C is Nil. By Lemma 3.7 the following are mutually opposite flags on V :

$$\{A^{d-i}V\}_{i=0}^d, \quad \{B^{d-i}V\}_{i=0}^d, \quad \{C^{d-i}V\}_{i=0}^d. \quad (45)$$

Lemma 13.15. *Let A, B, C denote an LR triple on V . In each row of the table below, we display a decomposition of V along with its induced flag on V :*

decomp. of V	induced flag on V
(A, B)	$\{A^{d-i}V\}_{i=0}^d$
(B, C)	$\{B^{d-i}V\}_{i=0}^d$
(C, A)	$\{C^{d-i}V\}_{i=0}^d$
(B, A)	$\{B^{d-i}V\}_{i=0}^d$
(C, B)	$\{C^{d-i}V\}_{i=0}^d$
(A, C)	$\{A^{d-i}V\}_{i=0}^d$

Proof. By Lemma 3.7. □

Lemma 13.16. *Let A, B, C denote an LR triple on V . In each row of the table below, we display a decomposition of V along with its dual decomposition of V^* .*

decomp. of V	dual decomp. of V^*
(A, B)	(\tilde{B}, \tilde{A})
(B, C)	(\tilde{C}, \tilde{B})
(C, A)	(\tilde{A}, \tilde{C})
(B, A)	(\tilde{A}, \tilde{B})
(C, B)	(\tilde{B}, \tilde{C})
(A, C)	(\tilde{C}, \tilde{A})

Proof. By Lemma 5.4. □

Lemma 13.17. *Let A, B, C denote an LR triple on V . In each row of the table below, we display a flag on V along with its dual flag on V^* .*

flag on V	dual flag on V^*
$\{A^{d-i}V\}_{i=0}^d$	$\{\tilde{A}^{d-i}V^*\}_{i=0}^d$
$\{B^{d-i}V\}_{i=0}^d$	$\{\tilde{B}^{d-i}V^*\}_{i=0}^d$
$\{C^{d-i}V\}_{i=0}^d$	$\{\tilde{C}^{d-i}V^*\}_{i=0}^d$

Proof. By Lemma 5.3. □

Lemma 13.18. *Let A, B, C denote an LR triple on V . In the table below we describe the action of A, B, C on the flags (45).*

flag on V	lowered by	raised by
$\{A^{d-i}V\}_{i=0}^d$	A	B, C
$\{B^{d-i}V\}_{i=0}^d$	B	C, A
$\{C^{d-i}V\}_{i=0}^d$	C	A, B

Proof. Any two of A, B, C form an LR pair on V . Apply Lemma 3.44 to these pairs. \square

Lemma 13.19. *Let A, B, C denote an LR triple on V . In each row of the table below, we display a decomposition $\{V_i\}_{i=0}^d$ of V . For $0 \leq i \leq d$ we give the action of A, B, C on V_i .*

dec. $\{V_i\}_{i=0}^d$	action of A on V_i	action of B on V_i	action of C on V_i
(A, B)	$AV_i = V_{i-1}$	$BV_i = V_{i+1}$	$CV_i \subseteq V_{i-1} + V_i + V_{i+1}$
(B, C)	$AV_i \subseteq V_{i-1} + V_i + V_{i+1}$	$BV_i = V_{i-1}$	$CV_i = V_{i+1}$
(C, A)	$AV_i = V_{i+1}$	$BV_i \subseteq V_{i-1} + V_i + V_{i+1}$	$CV_i = V_{i-1}$
(B, A)	$AV_i = V_{i+1}$	$BV_i = V_{i-1}$	$CV_i \subseteq V_{i-1} + V_i + V_{i+1}$
(C, B)	$AV_i \subseteq V_{i-1} + V_i + V_{i+1}$	$BV_i = V_{i+1}$	$CV_i = V_{i-1}$
(A, C)	$AV_i = V_{i-1}$	$BV_i \subseteq V_{i-1} + V_i + V_{i+1}$	$CV_i = V_{i+1}$

Proof. We verify the first row; the other rows are similarly verified. Let i be given. By construction $AV_i = V_{i-1}$ and $BV_i = V_{i+1}$. We now compute CV_i . By Lemma 13.15 the flag $\{A^{d-j}V\}_{j=0}^d$ is induced by $\{V_j\}_{j=0}^d$. By Lemma 13.18 the flag $\{A^{d-j}V\}_{j=0}^d$ is raised by C . By these comments and Definition 3.42,

$$CV_i \subseteq C(V_0 + \cdots + V_i) \subseteq V_0 + \cdots + V_{i+1}. \quad (46)$$

Similarly, the flag $\{B^{d-j}V\}_{j=0}^d$ is induced by $\{V_{d-j}\}_{j=0}^d$ and raised by C . Therefore

$$CV_i \subseteq C(V_i + \cdots + V_d) \subseteq V_{i-1} + \cdots + V_d. \quad (47)$$

Combining (46), (47) we obtain $CV_i \subseteq V_{i-1} + V_i + V_{i+1}$. \square

Lemma 13.20. *Let A, B, C denote an LR triple on V . Then the \mathbb{F} -algebra $\text{End}(V)$ is generated by any two of A, B, C .*

Proof. By Corollary 3.11. \square

Definition 13.21. Let A, B, C denote an LR triple on V . Recall that the pair A, B (resp. B, C) (resp. C, A) is an LR pair on V . For these LR pairs the idempotent sequence from Definition 3.5 is denoted as follows:

LR pair	idempotent sequence
A, B	$\{E_i\}_{i=0}^d$
B, C	$\{E'_i\}_{i=0}^d$
C, A	$\{E''_i\}_{i=0}^d$

We call the sequence

$$(\{E_i\}_{i=0}^d; \{E'_i\}_{i=0}^d; \{E''_i\}_{i=0}^d) \quad (48)$$

the *idempotent data* of A, B, C .

Lemma 13.22. *Let A, B, C denote an LR triple on V . Let α, β, γ denote nonzero scalars in \mathbb{F} . Then the idempotent data of $\alpha A, \beta B, \gamma C$ is equal to the idempotent data of A, B, C .*

Proof. By the last assertion of Lemma 3.8. \square

Let A, B, C denote an LR triple on V . Our next goal is to compute the idempotent data for the relatives of A, B, C .

Lemma 13.23. *Let A, B, C denote an LR triple on V , with idempotent data (48). In each row of the table below, we display an LR triple on V along with its idempotent data.*

LR triple	idempotent data
A, B, C	$(\{E_i\}_{i=0}^d; \{E'_i\}_{i=0}^d; \{E''_i\}_{i=0}^d)$
B, C, A	$(\{E'_i\}_{i=0}^d; \{E''_i\}_{i=0}^d; \{E_i\}_{i=0}^d)$
C, A, B	$(\{E''_i\}_{i=0}^d; \{E_i\}_{i=0}^d; \{E'_i\}_{i=0}^d)$
C, B, A	$(\{E'_{d-i}\}_{i=0}^d; \{E_{d-i}\}_{i=0}^d; \{E''_{d-i}\}_{i=0}^d)$
A, C, B	$(\{E''_{d-i}\}_{i=0}^d; \{E'_{d-i}\}_{i=0}^d; \{E_{d-i}\}_{i=0}^d)$
B, A, C	$(\{E_{d-i}\}_{i=0}^d; \{E''_{d-i}\}_{i=0}^d; \{E'_{d-i}\}_{i=0}^d)$

Proof. Use Lemma 3.6. \square

Lemma 13.24. *Let A, B, C denote an LR triple on V , with idempotent data (48). In each row of the table below, we display an LR triple on V^* along with its idempotent data.*

LR triple	idempotent data
$\tilde{A}, \tilde{B}, \tilde{C}$	$(\{E_{d-i}\}_{i=0}^d; \{E'_{d-i}\}_{i=0}^d; \{E''_{d-i}\}_{i=0}^d)$
$\tilde{B}, \tilde{C}, \tilde{A}$	$(\{\tilde{E}'_{d-i}\}_{i=0}^d; \{\tilde{E}''_{d-i}\}_{i=0}^d; \{\tilde{E}_{d-i}\}_{i=0}^d)$
$\tilde{C}, \tilde{A}, \tilde{B}$	$(\{\tilde{E}''_{d-i}\}_{i=0}^d; \{\tilde{E}_{d-i}\}_{i=0}^d; \{\tilde{E}'_{d-i}\}_{i=0}^d)$
$\tilde{C}, \tilde{B}, \tilde{A}$	$(\{\tilde{E}'_i\}_{i=0}^d; \{\tilde{E}_i\}_{i=0}^d; \{\tilde{E}''_i\}_{i=0}^d)$
$\tilde{A}, \tilde{C}, \tilde{B}$	$(\{\tilde{E}''_i\}_{i=0}^d; \{\tilde{E}'_i\}_{i=0}^d; \{\tilde{E}_i\}_{i=0}^d)$
$\tilde{B}, \tilde{A}, \tilde{C}$	$(\{\tilde{E}_i\}_{i=0}^d; \{\tilde{E}''_i\}_{i=0}^d; \{\tilde{E}'_i\}_{i=0}^d)$

Proof. By Lemmas 5.5, 13.23. \square

Lemma 13.25. *Let A, B, C denote an LR triple on V , with parameter array (44) and idempotent data (48). Then for $0 \leq i \leq d$,*

$$\begin{aligned}
E_i &= \frac{A^{d-i} B^d A^i}{\varphi_1 \cdots \varphi_d}, & E'_i &= \frac{B^{d-i} C^d B^i}{\varphi'_1 \cdots \varphi'_d}, & E''_i &= \frac{C^{d-i} A^d C^i}{\varphi''_1 \cdots \varphi''_d}, \\
E_i &= \frac{B^i A^d B^{d-i}}{\varphi_1 \cdots \varphi_d}, & E'_i &= \frac{C^i B^d C^{d-i}}{\varphi'_1 \cdots \varphi'_d}, & E''_i &= \frac{A^i C^d A^{d-i}}{\varphi''_1 \cdots \varphi''_d}.
\end{aligned}$$

Proof. By Lemma 3.18. \square

Lemma 13.26. *Let A, B, C denote an LR triple on V , with idempotent data (48). Then for $0 \leq i < j \leq d$ the following are zero:*

$$\begin{array}{cccc}
A^j E_i, & E_j A^{d-i}, & E_i B^j, & B^{d-i} E_j, \\
B^j E'_i, & E'_j B^{d-i}, & E'_i C^j, & C^{d-i} E'_j, \\
C^j E''_i, & E''_j C^{d-i}, & E''_i A^j, & A^{d-i} E''_j.
\end{array}$$

Proof. By Lemma 3.19. □

Lemma 13.27. *Let A, B, C denote an LR triple on V , with idempotent data (48). Then the following (i), (ii) hold for $0 \leq i, j \leq d$.*

(i) *Suppose $i + j < d$. Then*

$$E_i E'_j = 0, \quad E'_i E''_j = 0, \quad E''_i E_j = 0.$$

(ii) *Suppose $i + j > d$. Then*

$$E'_j E_i = 0, \quad E''_j E'_i = 0, \quad E_j E''_i = 0.$$

Proof. (i) We show $E_i E'_j = 0$. The sequence $\{E'_r V\}_{r=0}^d$ is the (B, C) decomposition of V , which induces the flag $\{B^{d-r} V\}_{r=0}^d$ on V . By this and Lemma 13.26,

$$E_i E'_j V \subseteq E_i (E'_0 V + \cdots + E'_j V) = E_i B^{d-j} V = 0.$$

This shows that $E_i E'_j = 0$. The remaining assertions are similarly shown.

(ii) We show $E'_j E_i = 0$. The sequence $\{E_{d-r} V\}_{r=0}^d$ is the (B, A) decomposition of V , which induces the flag $\{B^{d-r} V\}_{r=0}^d$ on V . Now by Lemma 13.26,

$$E'_j E_i V \subseteq E'_j (E_i V + \cdots + E_d V) = E'_j B^i V = 0.$$

This shows that $E'_j E_i = 0$. The remaining assertions are similarly shown. □

Lemma 13.28. *Let A, B, C denote an LR triple on V , with idempotent data (48). Then for $0 \leq i, j \leq d$,*

$$\begin{aligned} E_i E'_j E_i &= \delta_{i+j,d} E_i, & E'_i E''_j E'_i &= \delta_{i+j,d} E'_i, & E''_i E_j E''_i &= \delta_{i+j,d} E''_i, \\ E_i E''_j E_i &= \delta_{i+j,d} E_i, & E'_i E_j E'_i &= \delta_{i+j,d} E'_i, & E''_i E'_j E''_i &= \delta_{i+j,d} E''_i. \end{aligned}$$

Proof. Consider the product $E_i E'_j E_i$. First assume that $i + j < d$. Then $E_i E'_j = 0$ by Lemma 13.27(i), so $E_i E'_j E_i = 0$. Next assume that $i + j > d$. Then $E'_j E_i = 0$ by Lemma 13.27(ii), so $E_i E'_j E_i = 0$. Next assume that $i + j = d$. By our results so far,

$$E_i E'_j E_i = \sum_{r=0}^d E_i E'_r E_i = E_i I E_i = E_i.$$

We have verified our assertion for the product $E_i E'_j E_i$; the remaining assertions are similarly verified. □

Lemma 13.29. *Let A, B, C denote an LR triple on V , with idempotent data (48). Then for $0 \leq i, j \leq d$ the products*

$$\begin{array}{ccc} E_i E'_j, & E'_i E''_j, & E''_i E_j, \\ E_i E''_j, & E'_i E_j, & E''_i E'_j \end{array}$$

have trace 0 if $i + j \neq d$ and trace 1 if $i + j = d$.

Proof. For each displayed equation in Lemma 13.28, take the trace of each side and simplify the result using $\text{tr}(KL) = \text{tr}(LK)$. \square

Proposition 13.30. *Let A, B, C denote an LR triple on V , with parameter array (44) and idempotent data (48). In the table below, for each map F in the header row, we display the trace of FE_i , FE'_i , FE''_i for $0 \leq i \leq d$.*

F	AB	BA	BC	CB	CA	AC
$\text{tr}(FE_i)$	φ_{i+1}	φ_i	φ'_{d-i+1}	φ'_{d-i}	φ''_{d-i+1}	φ''_{d-i}
$\text{tr}(FE'_i)$	φ_{d-i+1}	φ_{d-i}	φ'_{i+1}	φ'_i	φ''_{d-i+1}	φ''_{d-i}
$\text{tr}(FE''_i)$	φ_{d-i+1}	φ_{d-i}	φ'_{d-i+1}	φ'_{d-i}	φ''_{i+1}	φ''_i

Proof. We verify the first column of the table. We have $\text{tr}(ABE_i) = \varphi_{i+1}$ by Lemma 3.20. To verify $\text{tr}(ABE'_i) = \varphi_{d-i+1}$ and $\text{tr}(ABE''_i) = \varphi_{d-i+1}$, eliminate AB using the equation on the left in (3), and evaluate the result using Lemma 13.29. We have verified the first column of the table; the remaining columns are similarly verified. \square

In Proposition 13.30 we used the trace function to describe the parameter array of an LR triple. We now use the trace function to define some more parameters for an LR triple.

Definition 13.31. Let A, B, C denote an LR triple on V , with idempotent data (48). For $0 \leq i \leq d$ define

$$a_i = \text{tr}(CE_i), \quad a'_i = \text{tr}(AE'_i), \quad a''_i = \text{tr}(BE''_i). \quad (49)$$

We call the sequence

$$(\{a_i\}_{i=0}^d; \{a'_i\}_{i=0}^d; \{a''_i\}_{i=0}^d) \quad (50)$$

the *trace data* of A, B, C .

Our next goal is to describe the meaning of the trace data from several points of view.

Lemma 13.32. *Let A, B, C denote an LR triple on V , with trace data (50). Consider a basis for V that induces the (A, B) -decomposition (resp. (B, C) -decomposition) (resp. (C, A) -decomposition) of V . Then for $0 \leq i \leq d$, a_i (resp. a'_i) (resp. a''_i) is the (i, i) -entry of the matrix in $\text{Mat}_{d+1}(\mathbb{F})$ that represents C (resp. A) (resp. B) with respect to this basis.*

Proof. To obtain the assertion about a_i , in the equation on the left in (49) represent C and E_i by matrices with respect to the given basis. The other assertions are similarly obtained. \square

Lemma 13.33. *Let A, B, C denote an LR triple on V , with idempotent data (48) and trace data (50). Then for $0 \leq i \leq d$,*

$$E_i C E_i = a_i E_i, \quad E'_i A E'_i = a'_i E'_i, \quad E''_i B E''_i = a''_i E''_i.$$

Proof. We verify the equation on the left. Since E_i is idempotent and rank 1, there exists $a \in \mathbb{F}$ such that $E_i C E_i = a E_i$. In this equation, take the trace of each side and use Definition 13.31 to get $a = a_i$. \square

Lemma 13.34. *Let A, B, C denote an LR triple on V , with trace data (50). Then*

$$0 = \sum_{i=0}^d a_i, \quad 0 = \sum_{i=0}^d a'_i, \quad 0 = \sum_{i=0}^d a''_i.$$

Proof. The sum $\sum_{i=0}^d a_i$ is the trace of C , which is zero since C is nilpotent. The remaining assertions are similarly shown. \square

Lemma 13.35. *Let A, B, C denote an LR triple on V , with trace data (50). Let α, β, γ denote nonzero scalars in \mathbb{F} . Then the LR triple $\alpha A, \beta B, \gamma C$ has trace data*

$$(\{\gamma a_i\}_{i=0}^d; \{\alpha a'_i\}_{i=0}^d; \{\beta a''_i\}_{i=0}^d).$$

Proof. Use Lemma 13.22 and Definition 13.31. \square

Let A, B, C denote an LR triple on V . Our next goal is to compute the trace data for the relatives of A, B, C .

Lemma 13.36. *Let A, B, C denote an LR triple on V , with trace data (50). In each row of the table below, we display an LR triple on V along with its trace data.*

LR triple	trace data
A, B, C	$(\{a_i\}_{i=0}^d; \{a'_i\}_{i=0}^d; \{a''_i\}_{i=0}^d)$
B, C, A	$(\{a'_i\}_{i=0}^d; \{a''_i\}_{i=0}^d; \{a_i\}_{i=0}^d)$
C, A, B	$(\{a''_i\}_{i=0}^d; \{a_i\}_{i=0}^d; \{a'_i\}_{i=0}^d)$
C, B, A	$(\{a'_{d-i}\}_{i=0}^d; \{a_{d-i}\}_{i=0}^d; \{a''_{d-i}\}_{i=0}^d)$
A, C, B	$(\{a''_{d-i}\}_{i=0}^d; \{a'_{d-i}\}_{i=0}^d; \{a_{d-i}\}_{i=0}^d)$
B, A, C	$(\{a_{d-i}\}_{i=0}^d; \{a'_{d-i}\}_{i=0}^d; \{a''_{d-i}\}_{i=0}^d)$

Proof. By Lemma 13.23 and Definition 13.31. \square

Lemma 13.37. *Let A, B, C denote an LR triple on V , with trace data (50). In each row of the table below, we display an LR triple on V^* along with its trace data.*

LR triple	trace data
A, \tilde{B}, \tilde{C}	$(\{a_{d-i}\}_{i=0}^d; \{a'_{d-i}\}_{i=0}^d; \{a''_{d-i}\}_{i=0}^d)$
$\tilde{B}, \tilde{C}, \tilde{A}$	$(\{a'_{d-i}\}_{i=0}^d; \{a''_{d-i}\}_{i=0}^d; \{a_{d-i}\}_{i=0}^d)$
$\tilde{C}, \tilde{A}, \tilde{B}$	$(\{a''_{d-i}\}_{i=0}^d; \{a_{d-i}\}_{i=0}^d; \{a'_{d-i}\}_{i=0}^d)$
$\tilde{C}, \tilde{B}, \tilde{A}$	$(\{a'_i\}_{i=0}^d; \{a_i\}_{i=0}^d; \{a''_i\}_{i=0}^d)$
$\tilde{A}, \tilde{C}, \tilde{B}$	$(\{a''_i\}_{i=0}^d; \{a'_i\}_{i=0}^d; \{a_i\}_{i=0}^d)$
$\tilde{B}, \tilde{A}, \tilde{C}$	$(\{a_i\}_{i=0}^d; \{a''_i\}_{i=0}^d; \{a'_i\}_{i=0}^d)$

Proof. An element in $\text{End}(V)$ has the same trace as its adjoint. The result follows from this along with Lemma 13.24 and Definition 13.31. \square

Let A, B, C denote an LR triple on V , with parameter array (44), idempotent data (48), and trace data (50). Associated with A, B, C are 12 types of bases for V :

$$(A, B), \quad \text{inverted } (A, B), \quad (B, A), \quad \text{inverted } (B, A), \quad (51)$$

$$(B, C), \quad \text{inverted } (B, C), \quad (C, B), \quad \text{inverted } (C, B), \quad (52)$$

$$(C, A), \quad \text{inverted } (C, A), \quad (A, C), \quad \text{inverted } (A, C). \quad (53)$$

We now consider the actions of A, B, C on these bases. We will use the following notation.

Definition 13.38. For the above LR triple A, B, C consider the 12 types of bases for V from (51)–(53). For each type \natural in the list and $F \in \text{End}(V)$ let F^\natural denote the matrix in $\text{Mat}_{d+1}(\mathbb{F})$ that represents F with respect to a basis for V of type \natural . Note that the map $\natural : \text{End}(V) \rightarrow \text{Mat}_{d+1}(\mathbb{F})$, $F \mapsto F^\natural$ is an \mathbb{F} -algebra isomorphism.

Proposition 13.39. For the above LR triple A, B, C consider the 12 types of bases for V from (51)–(53). For each type \natural in the list, the entries of A^\natural , B^\natural , C^\natural are given in the table below. All entries not shown are zero.

\natural	$A_{i,i-1}^\natural$	$A_{i,i}^\natural$	$A_{i-1,i}^\natural$	$B_{i,i-1}^\natural$	$B_{i,i}^\natural$	$B_{i-1,i}^\natural$	$C_{i,i-1}^\natural$	$C_{i,i}^\natural$	$C_{i-1,i}^\natural$
(A, B)	0	0	1	φ_i	0	0	$\frac{\varphi_{d-i+1}''}{\varphi_i'}$	a_i	$\frac{\varphi_{d-i+1}'}{\varphi_i''}$
inv. (A, B)	1	0	0	0	0	φ_{d-i+1}	$\frac{\varphi_i'}{\varphi_{d-i+1}''}$	a_{d-i}	$\frac{\varphi_i''}{\varphi_{d-i+1}'}$
(B, A)	φ_{d-i+1}	0	0	0	0	1	$\frac{\varphi_i'}{\varphi_{d-i+1}''}$	a_{d-i}	$\frac{\varphi_i''}{\varphi_{d-i+1}'}$
inv. (B, A)	0	0	φ_i	1	0	0	$\frac{\varphi_{d-i+1}''}{\varphi_i'}$	a_i	$\frac{\varphi_{d-i+1}'}{\varphi_i''}$
(B, C)	φ_{d-i+1}	a_i'	$\frac{\varphi_{d-i+1}''}{\varphi_i'}$	0	0	1	φ_i'	0	0
inv. (B, C)	$\frac{\varphi_i''}{\varphi_{d-i+1}'}$	a_{d-i}'	φ_i	1	0	0	0	0	φ_{d-i+1}'
(C, B)	$\frac{\varphi_i''}{\varphi_{d-i+1}'}$	a_{d-i}'	$\frac{\varphi_i}{\varphi_{d-i+1}'}$	φ_{d-i+1}'	0	0	0	0	1
inv. (C, B)	$\frac{\varphi_{d-i+1}''}{\varphi_i'}$	a_i'	φ_{d-i+1}'	0	0	φ_i'	1	0	0
(C, A)	φ_i''	0	0	φ_{d-i+1}'	a_i''	$\frac{\varphi_{d-i+1}''}{\varphi_i'}$	0	0	1
inv. (C, A)	0	0	φ_{d-i+1}''	$\frac{\varphi_i}{\varphi_{d-i+1}'}$	a_{d-i}''	φ_i'	1	0	0
(A, C)	0	0	1	φ_i	a_{d-i}''	$\frac{\varphi_i'}{\varphi_{d-i+1}'}$	φ_{d-i+1}''	0	0
inv. (A, C)	1	0	0	$\frac{\varphi_{d-i+1}''}{\varphi_i'}$	a_i''	φ_{d-i+1}'	0	0	φ_i''

Proof. We verify the first row of the table. Consider a basis for V of type $\natural = (A, B)$. The entries of A^\natural and B^\natural are given in Lemma 3.23. We now compute the entries of C^\natural . This matrix is tridiagonal by Lemma 13.19. The diagonal entries of C^\natural are given in Lemma 13.32. As we compute additional entries of C^\natural , we will use the fact that for $0 \leq j \leq d$ the matrix E_j^\natural has (j, j) -entry 1 and all other entries 0. For $1 \leq i \leq d$ we now compute the $(i, i-1)$ -entry of C^\natural . We evaluate $\text{tr}(CAE_i)$ in two ways. On one hand, by Proposition 13.30 this trace is equal to φ_{d-i+1}'' . On the other hand, by linear algebra this trace is equal to $\text{tr}(C^\natural A^\natural E_i^\natural)$, which is equal to the (i, i) -entry of $C^\natural A^\natural$ by the form of E_i^\natural . By the form of A^\natural , the (i, i) -entry of $C^\natural A^\natural$ is equal to $C_{i,i-1}^\natural$. By these comments $C_{i,i-1}^\natural = \varphi_{d-i+1}''$. Next we compute the

$(i-1, i)$ -entry of C^\natural . We evaluate $\text{tr}(BCE_i)$ in two ways. On one hand, by Proposition 13.30 this trace is equal to φ'_{d-i+1} . On the other hand, by linear algebra this trace is equal to $\text{tr}(B^\natural C^\natural E_i^\natural)$, which is equal to the (i, i) -entry of $B^\natural C^\natural$ by the form of E_i^\natural . By the form of B^\natural , the (i, i) -entry of $B^\natural C^\natural$ is equal to $\varphi_i C_{i-1, i}^\natural$. By these comments $C_{i-1, i}^\natural = \varphi'_{d-i+1}/\varphi_i$. We have verified the first row of the table, and the remaining rows are similarly verified. \square

Proposition 13.40. *An LR triple is uniquely determined up to isomorphism by its parameter array and trace data.*

Proof. In Proposition 13.39, the matrix entries are determined by the parameter array and trace data. \square

Let A, B, C denote an LR triple on V . Recall the 12 types of bases for V from (51)–(53). We now consider how these bases are related. As we proceed, keep in mind that any permutation of A, B, C is an LR triple on V .

Lemma 13.41. *Let A, B, C denote an LR triple on V . Let $\{u_i\}_{i=0}^d$ denote an (A, C) -basis of V , and let $\{v_i\}_{i=0}^d$ denote an (A, B) -basis of V . Then the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$ is upper triangular and Toeplitz.*

Proof. Let $S \in \text{Mat}_{d+1}(\mathbb{F})$ denote the transition matrix in question. By Definition 3.21 and the construction, $Au_i = u_{i-1}$ ($1 \leq i \leq d$), $Au_0 = 0$, $Av_i = v_{i-1}$ ($1 \leq i \leq d$), $Av_0 = 0$. Now by Proposition 12.8, S is upper triangular and Toeplitz. \square

Definition 13.42. Two bases of V will be called *compatible* whenever the transition matrix from one basis to the other is upper triangular and Toeplitz, with all diagonal entries 1.

Lemma 13.43. *Let A, B, C denote an LR triple on V . Given an (A, C) -basis of V , there exists a compatible (A, B) -basis of V .*

Proof. Let $\{u_i\}_{i=0}^d$ denote the (A, C) -basis in question. Let $\{v_i\}_{i=0}^d$ denote an (A, B) -basis of V . Let $S \in \text{Mat}_{d+1}(\mathbb{F})$ denote the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$. By construction S is invertible. By Lemma 13.41, S is upper triangular and Toeplitz. Let $\{\alpha_i\}_{i=0}^d$ denote the corresponding parameters, and note that $\alpha_0 \neq 0$. Define $v'_i = v_i/\alpha_0$ for $0 \leq i \leq d$. Then $\{v'_i\}_{i=0}^d$ is an (A, B) -basis of V . The transition matrix from $\{u_i\}_{i=0}^d$ to $\{v'_i\}_{i=0}^d$ is S/α_0 . This matrix is upper triangular and Toeplitz, with all diagonal entries 1. Now by Definition 13.42 the basis $\{v'_i\}_{i=0}^d$ is compatible with $\{u_i\}_{i=0}^d$. \square

Definition 13.44. Let A, B, C denote an LR triple on V . We define matrices T, T', T'' in $\text{Mat}_{d+1}(\mathbb{F})$ as follows:

- (i) T is the transition matrix from a (C, B) -basis of V to a compatible (C, A) -basis of V ;
- (ii) T' is the transition matrix from an (A, C) -basis of V to a compatible (A, B) -basis of V ;
- (iii) T'' is the transition matrix from a (B, A) -basis of V to a compatible (B, C) -basis of V .

Definition 13.45. Let A, B, C denote an LR triple on V . By Definition 13.42 the associated matrix T (resp. T') (resp. T'') is upper triangular and Toeplitz; let $\{\alpha_i\}_{i=0}^d$ (resp. $\{\alpha'_i\}_{i=0}^d$) (resp. $\{\alpha''_i\}_{i=0}^d$) denote the corresponding parameters. Let $\{\beta_i\}_{i=0}^d$, $\{\beta'_i\}_{i=0}^d$, $\{\beta''_i\}_{i=0}^d$ denote the parameters for T^{-1} , $(T')^{-1}$, $(T'')^{-1}$ respectively. We call the 6-tuple

$$(\{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^d; \{\alpha'_i\}_{i=0}^d, \{\beta'_i\}_{i=0}^d; \{\alpha''_i\}_{i=0}^d, \{\beta''_i\}_{i=0}^d) \quad (54)$$

the *Toeplitz data* for A, B, C . For notational convenience define each of the following to be zero:

$$\alpha_{d+1}, \quad \alpha'_{d+1}, \quad \alpha''_{d+1}, \quad \beta_{d+1}, \quad \beta'_{d+1}, \quad \beta''_{d+1}.$$

Lemma 13.46. Referring to Definition 13.45,

$$\alpha_0 = 1, \quad \alpha'_0 = 1, \quad \alpha''_0 = 1, \quad (55)$$

$$\beta_0 = 1, \quad \beta'_0 = 1, \quad \beta''_0 = 1. \quad (56)$$

Moreover

$$\beta_1 = -\alpha_1, \quad \beta'_1 = -\alpha'_1, \quad \beta''_1 = -\alpha''_1. \quad (57)$$

Proof. Concerning (55), (56) the matrices T, T', T'' and their inverses are all transition matrices between a pair of compatible bases. So their diagonal entries are all 1 by Definition 13.42. Line (57) comes from above Lemma 12.5. \square

Lemma 13.47. Referring to Definitions 3.49, 13.44,

$$\begin{aligned} T &= \sum_{i=0}^d \alpha_i \tau^i & T' &= \sum_{i=0}^d \alpha'_i \tau^i & T'' &= \sum_{i=0}^d \alpha''_i \tau^i, \\ T^{-1} &= \sum_{i=0}^d \beta_i \tau^i, & (T')^{-1} &= \sum_{i=0}^d \beta'_i \tau^i, & (T'')^{-1} &= \sum_{i=0}^d \beta''_i \tau^i. \end{aligned}$$

Moreover T, T', T'', τ mutually commute.

Proof. By Lemma 12.3. \square

Lemma 13.48. Let A, B, C denote an LR triple on V , with Toeplitz data (54). Let α, β, γ denote nonzero scalars in \mathbb{F} . Then the LR triple $\alpha A, \beta B, \gamma C$ has Toeplitz data

$$(\{\gamma^{-i} \alpha_i\}_{i=0}^d, \{\gamma^{-i} \beta_i\}_{i=0}^d; \{\alpha^{-i} \alpha'_i\}_{i=0}^d, \{\alpha^{-i} \beta'_i\}_{i=0}^d; \{\beta^{-i} \alpha''_i\}_{i=0}^d, \{\beta^{-i} \beta''_i\}_{i=0}^d). \quad (58)$$

Proof. Use Lemma 12.9(ii). \square

Let A, B, C denote an LR triple on V . Our next goal is to compute the Toeplitz data for the relatives of A, B, C .

Lemma 13.49. *Let A, B, C denote an LR triple on V , with Toeplitz data (54). In each row of the table below, we display an LR triple on V along with its Toeplitz data.*

LR triple	Toeplitz data
A, B, C	$(\{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^d; \{\alpha'_i\}_{i=0}^d, \{\beta'_i\}_{i=0}^d; \{\alpha''_i\}_{i=0}^d, \{\beta''_i\}_{i=0}^d)$
B, C, A	$(\{\alpha'_i\}_{i=0}^d, \{\beta'_i\}_{i=0}^d; \{\alpha''_i\}_{i=0}^d, \{\beta''_i\}_{i=0}^d; \{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^d)$
C, A, B	$(\{\alpha''_i\}_{i=0}^d, \{\beta''_i\}_{i=0}^d; \{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^d; \{\alpha'_i\}_{i=0}^d, \{\beta'_i\}_{i=0}^d)$
C, B, A	$(\{\beta'_i\}_{i=0}^d, \{\alpha'_i\}_{i=0}^d; \{\beta_i\}_{i=0}^d, \{\alpha_i\}_{i=0}^d; \{\beta''_i\}_{i=0}^d, \{\alpha''_i\}_{i=0}^d)$
A, C, B	$(\{\beta''_i\}_{i=0}^d, \{\alpha''_i\}_{i=0}^d; \{\beta'_i\}_{i=0}^d, \{\alpha'_i\}_{i=0}^d; \{\beta_i\}_{i=0}^d, \{\alpha_i\}_{i=0}^d)$
B, A, C	$(\{\beta_i\}_{i=0}^d, \{\alpha_i\}_{i=0}^d; \{\beta'_i\}_{i=0}^d, \{\alpha'_i\}_{i=0}^d; \{\beta''_i\}_{i=0}^d, \{\alpha''_i\}_{i=0}^d)$

Proof. By Definition 13.44 and the construction. \square

Lemma 13.50. *Let $\{u_i\}_{i=0}^d$ (resp. $\{v_i\}_{i=0}^d$) denote a basis of V , and let $\{u'_i\}_{i=0}^d$ (resp. $\{v'_i\}_{i=0}^d$) denote the dual basis of V^* . Then the following are equivalent:*

- (i) $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$ are compatible;
- (ii) $\{u'_{d-i}\}_{i=0}^d$ and $\{v'_{d-i}\}_{i=0}^d$ are compatible.

Moreover, suppose (i), (ii) hold. Then the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$ is the inverse of the transition matrix from $\{u'_{d-i}\}_{i=0}^d$ to $\{v'_{d-i}\}_{i=0}^d$.

Proof. Let $S \in \text{Mat}_{d+1}(\mathbb{F})$ denote the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$. Then S^t is the transition matrix from $\{v'_i\}_{i=0}^d$ to $\{u'_i\}_{i=0}^d$. Then $(S^t)^{-1}$ is the transition matrix from $\{u'_i\}_{i=0}^d$ to $\{v'_i\}_{i=0}^d$. Then $\mathbf{Z}(S^t)^{-1}\mathbf{Z}$ is the transition matrix from $\{u'_{d-i}\}_{i=0}^d$ to $\{v'_{d-i}\}_{i=0}^d$. Note that S is upper triangular and Toeplitz with all diagonal entries 1, if and only if S^{-1} is upper triangular and Toeplitz with all diagonal entries 1. By this and Lemma 12.4, we see that S is upper triangular and Toeplitz with all diagonal entries 1, if and only if $\mathbf{Z}(S^t)^{-1}\mathbf{Z}$ is upper triangular and Toeplitz with all diagonal entries 1, and in this case $\mathbf{Z}(S^t)^{-1}\mathbf{Z} = S^{-1}$. The result follows. \square

Lemma 13.51. *Let A, B, C denote an LR triple on V , with Toeplitz data (54). In each row of the table below, we display an LR triple on V^* along with its Toeplitz data.*

LR triple	Toeplitz data
$\tilde{A}, \tilde{B}, \tilde{C}$	$(\{\beta_i\}_{i=0}^d, \{\alpha_i\}_{i=0}^d; \{\beta'_i\}_{i=0}^d, \{\alpha'_i\}_{i=0}^d; \{\beta''_i\}_{i=0}^d, \{\alpha''_i\}_{i=0}^d)$
$\tilde{B}, \tilde{C}, \tilde{A}$	$(\{\beta'_i\}_{i=0}^d, \{\alpha'_i\}_{i=0}^d; \{\beta''_i\}_{i=0}^d, \{\alpha''_i\}_{i=0}^d; \{\beta_i\}_{i=0}^d, \{\alpha_i\}_{i=0}^d)$
$\tilde{C}, \tilde{A}, \tilde{B}$	$(\{\beta''_i\}_{i=0}^d, \{\alpha''_i\}_{i=0}^d; \{\beta_i\}_{i=0}^d, \{\alpha_i\}_{i=0}^d; \{\beta'_i\}_{i=0}^d, \{\alpha'_i\}_{i=0}^d)$
$\tilde{C}, \tilde{B}, \tilde{A}$	$(\{\alpha'_i\}_{i=0}^d, \{\beta'_i\}_{i=0}^d; \{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^d; \{\alpha''_i\}_{i=0}^d, \{\beta''_i\}_{i=0}^d)$
$\tilde{A}, \tilde{C}, \tilde{B}$	$(\{\alpha''_i\}_{i=0}^d, \{\beta''_i\}_{i=0}^d; \{\alpha'_i\}_{i=0}^d, \{\beta'_i\}_{i=0}^d; \{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^d)$
$\tilde{B}, \tilde{A}, \tilde{C}$	$(\{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^d; \{\alpha''_i\}_{i=0}^d, \{\beta''_i\}_{i=0}^d; \{\alpha'_i\}_{i=0}^d, \{\beta'_i\}_{i=0}^d)$

Proof. Use Lemma 5.7, Definition 13.44, and Lemma 13.50. \square

Until further notice fix an LR triple A, B, C on V , with parameter array (44), idempotent data (48), trace data (50), and Toeplitz data (54). Recall the 12 types of bases for V from (51)–(53). As we consider how these bases are related, it is convenient to work with specific bases of each type. Fix nonzero vectors

$$\eta \in A^d V, \quad \eta' \in B^d V, \quad \eta'' \in C^d V, \quad (59)$$

$$\tilde{\eta} \in \tilde{A}^d V^*, \quad \tilde{\eta}' \in \tilde{B}^d V^*, \quad \tilde{\eta}'' \in \tilde{C}^d V^*. \quad (60)$$

By construction,

$$A\eta = 0, \quad B\eta' = 0, \quad C\eta'' = 0, \quad (61)$$

$$\tilde{A}\tilde{\eta} = 0, \quad \tilde{B}\tilde{\eta}' = 0, \quad \tilde{C}\tilde{\eta}'' = 0. \quad (62)$$

We mention a result for later use.

Lemma 13.52. *The following scalars are nonzero:*

$$(\eta, \tilde{\eta}'), \quad (\eta', \tilde{\eta}''), \quad (\eta'', \tilde{\eta}), \quad (63)$$

$$(\eta, \tilde{\eta}''), \quad (\eta', \tilde{\eta}), \quad (\eta'', \tilde{\eta}'). \quad (64)$$

For $d \geq 1$ the following scalars are zero:

$$(\eta, \tilde{\eta}), \quad (\eta', \tilde{\eta}'), \quad (\eta'', \tilde{\eta}''). \quad (65)$$

Proof. We show that $(\eta, \tilde{\eta}') \neq 0$. By assumption $0 \neq \eta \in A^d V$ and $0 \neq \tilde{\eta}' \in \tilde{B}^d V^*$. The flags $\{B^{d-i}V\}_{i=0}^d$ and $\{\tilde{B}^{d-i}V^*\}_{i=0}^d$ are dual by Lemma 13.17; therefore BV is the orthogonal complement of $\tilde{B}^d V^*$. The flags $\{A^{d-i}V\}_{i=0}^d$ and $\{B^{d-i}V\}_{i=0}^d$ are opposite; therefore $A^d V \cap BV = 0$. By these comments $(\eta, \tilde{\eta}') \neq 0$. The other five inner products in (63), (64) are similarly shown to be nonzero. Next assume that $d \geq 1$. We show that $(\eta, \tilde{\eta}) = 0$. By construction $\eta \in A^d V$ and $\tilde{\eta} \in \tilde{A}^d V^*$. The flags $\{A^{d-i}V\}_{i=0}^d$ and $\{\tilde{A}^{d-i}V^*\}_{i=0}^d$ are dual by Lemma 13.17; therefore AV is the orthogonal complement of $\tilde{A}^d V^*$. The subspace AV contains $A^d V$ since $d \geq 1$; therefore $A^d V$ is orthogonal to $\tilde{A}^d V^*$. By these comments $(\eta, \tilde{\eta}) = 0$. The other two inner products in (65) are similarly shown to be zero. \square

We now display some bases for V of the types (51)–(53).

Lemma 13.53. *In each row of the tables below, for $0 \leq i \leq d$ we display a vector $v_i \in V$. The vectors $\{v_i\}_{i=0}^d$ form a basis for V ; we give the type and the induced decomposition of V .*

v_i	type of basis	induced dec. of V
$B^i \eta$	inverted (B, A)	(A, B)
$B^{d-i} \eta$	(B, A)	(B, A)
$(\varphi_1 \cdots \varphi_i)^{-1} B^i \eta$	(A, B)	(A, B)
$(\varphi_1 \cdots \varphi_{d-i})^{-1} B^{d-i} \eta$	inverted (A, B)	(B, A)
$C^i \eta$	inverted (C, A)	(A, C)
$C^{d-i} \eta$	(C, A)	(C, A)
$(\varphi''_d \cdots \varphi''_{d-i+1})^{-1} C^i \eta$	(A, C)	(A, C)
$(\varphi''_d \cdots \varphi''_{i+1})^{-1} C^{d-i} \eta$	inverted (A, C)	(C, A)

v_i	type of basis	induced dec. of V
$C^i \eta'$	inverted (C, B)	(B, C)
$C^{d-i} \eta'$	(C, B)	(C, B)
$(\varphi'_1 \cdots \varphi'_i)^{-1} C^i \eta'$	(B, C)	(B, C)
$(\varphi'_1 \cdots \varphi'_{d-i})^{-1} C^{d-i} \eta'$	inverted (B, C)	(C, B)
$A^i \eta'$	inverted (A, B)	(B, A)
$A^{d-i} \eta'$	(A, B)	(A, B)
$(\varphi_d \cdots \varphi_{d-i+1})^{-1} A^i \eta'$	(B, A)	(B, A)
$(\varphi_d \cdots \varphi_{i+1})^{-1} A^{d-i} \eta'$	inverted (B, A)	(A, B)

v_i	type of basis	induced dec. of V
$A^i \eta''$	inverted (A, C)	(C, A)
$A^{d-i} \eta''$	(A, C)	(A, C)
$(\varphi''_1 \cdots \varphi''_i)^{-1} A^i \eta''$	(C, A)	(C, A)
$(\varphi''_1 \cdots \varphi''_{d-i})^{-1} A^{d-i} \eta''$	inverted (C, A)	(A, C)
$B^i \eta''$	inverted (B, C)	(C, B)
$B^{d-i} \eta''$	(B, C)	(B, C)
$(\varphi'_d \cdots \varphi'_{d-i+1})^{-1} B^i \eta''$	(C, B)	(C, B)
$(\varphi'_d \cdots \varphi'_{i+1})^{-1} B^{d-i} \eta''$	inverted (C, B)	(B, C)

Proof. Use Lemmas 3.24, 3.27, 3.30, 3.33. □

Lemma 13.54. *In each of (i)–(iii) below we describe two bases from the tables in Lemma 13.53. These two bases are compatible.*

- (i) *the (A, B) -basis in the first table, and the (A, C) -basis in the first table;*
- (ii) *the (B, C) -basis in the second table, and the (B, A) -basis in the second table;*
- (iii) *the (C, A) -basis in the third table, and the (C, B) -basis in the third table.*

Proof. (i) Let $\{u_i\}_{i=0}^d$ denote the (A, C) -basis in the first table, and let $\{v_i\}_{i=0}^d$ denote the (A, B) -basis in the first table. Let $S \in \text{Mat}_{d+1}(\mathbb{F})$ denote the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$. By Lemma 13.41, S is upper triangular and Toeplitz; let $\{s_i\}_{i=0}^d$ denote the corresponding parameters. Note that u_0 and v_0 are both equal to η ; therefore $s_0 = 1$. Consequently the diagonal entries of S are all 1. The bases $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$ are compatible by Definition 13.42.

(ii), (iii) Similar to the proof of (i) above. □

Lemma 13.55. *Referring to Lemma 13.53,*

- (i) *T is the transition matrix from the (C, B) -basis in the third table, to the (C, A) -basis in the third table;*
- (ii) *T' is the transition matrix from the (A, C) -basis in the first table, to the (A, B) -basis in the first table;*

(iii) T'' is the transition matrix from the (B, A) -basis in the second table, to the (B, C) -basis in the second table.

Proof. By Definition 13.44 and Lemma 13.54. \square

Lemma 13.56. For $0 \leq j \leq d$,

$$\begin{aligned} B^j \eta &= \sum_{i=0}^j \alpha'_{j-i} \frac{\varphi_1 \cdots \varphi_j}{\varphi_d'' \cdots \varphi_{d-i+1}''} C^i \eta, & C^j \eta &= \sum_{i=0}^j \beta'_{j-i} \frac{\varphi_d'' \cdots \varphi_{d-j+1}''}{\varphi_1 \cdots \varphi_i} B^i \eta, \\ C^j \eta' &= \sum_{i=0}^j \alpha''_{j-i} \frac{\varphi_1' \cdots \varphi_j'}{\varphi_d \cdots \varphi_{d-i+1}} A^i \eta', & A^j \eta' &= \sum_{i=0}^j \beta''_{j-i} \frac{\varphi_d \cdots \varphi_{d-j+1}}{\varphi_1' \cdots \varphi_i'} C^i \eta', \\ A^j \eta'' &= \sum_{i=0}^j \alpha_{j-i} \frac{\varphi_1'' \cdots \varphi_j''}{\varphi_d' \cdots \varphi_{d-i+1}'} B^i \eta'', & B^j \eta'' &= \sum_{i=0}^j \beta_{j-i} \frac{\varphi_d' \cdots \varphi_{d-j+1}'}{\varphi_1'' \cdots \varphi_i''} A^i \eta''. \end{aligned}$$

Proof. Use Lemmas 13.53, 13.55. \square

Lemma 13.57. We have

$$\eta = \frac{(\eta, \tilde{\eta}'')}{(\eta', \tilde{\eta}'')} \sum_{i=0}^d \alpha_i C^i \eta', \quad \eta = \frac{(\eta, \tilde{\eta}')}{(\eta'', \tilde{\eta}')} \sum_{i=0}^d \beta_i'' B^i \eta'', \quad (66)$$

$$\eta' = \frac{(\eta', \tilde{\eta})}{(\eta'', \tilde{\eta})} \sum_{i=0}^d \alpha_i' A^i \eta'', \quad \eta' = \frac{(\eta', \tilde{\eta}'')}{(\eta, \tilde{\eta}'')} \sum_{i=0}^d \beta_i C^i \eta, \quad (67)$$

$$\eta'' = \frac{(\eta'', \tilde{\eta}')}{(\eta, \tilde{\eta}')} \sum_{i=0}^d \alpha_i'' B^i \eta, \quad \eta'' = \frac{(\eta'', \tilde{\eta})}{(\eta', \tilde{\eta})} \sum_{i=0}^d \beta_i' A^i \eta'. \quad (68)$$

Proof. We verify the equation on the left in (66). By Lemma 13.53, $\{C^{d-i} \eta'\}_{i=0}^d$ is a (C, B) -basis of V . Let $\{v_i\}_{i=0}^d$ denote a compatible (C, A) -basis of V . By Definition 13.44(i) T is the transition matrix from $\{C^{d-i} \eta'\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$. By the discussion in Definition 13.45, T is upper triangular and Toeplitz with parameters $\{\alpha_i\}_{i=0}^d$. By these comments $v_d = \sum_{i=0}^d \alpha_i C^i \eta'$. By construction v_d is a basis of $A^d V$. Since η is also a basis of $A^d V$, there exists $0 \neq \zeta \in \mathbb{F}$ such that $\eta = \zeta v_d$. Therefore

$$\eta = \zeta \sum_{i=0}^d \alpha_i C^i \eta'. \quad (69)$$

We now compute ζ . In the equation (69), take the inner product of each side with $\tilde{\eta}''$. By Lemma 13.16 (row 2 of the table) we have $(C^i \eta', \tilde{\eta}'') = 0$ for $1 \leq i \leq d$. By this and $\alpha_0 = 1$ we obtain $(\eta, \tilde{\eta}'') = \zeta(\eta', \tilde{\eta}'')$. Therefore

$$\zeta = \frac{(\eta, \tilde{\eta}'')}{(\eta', \tilde{\eta}'')}. \quad (70)$$

Combining (69), (70) we obtain the equation on the left in (66). The remaining equations in (66)–(68) are similarly verified. \square

In the next two lemmas we give some results about α_1 . Similar results hold for α'_1, α''_1 .

Lemma 13.58. *Assume $d \geq 1$. Then the vector η'' is an eigenvector for $A/\varphi''_1 - B/\varphi'_d$ with eigenvalue α_1 .*

Proof. In Lemma 13.56, set $j = 1$ in either equation from the third row. \square

Lemma 13.59. *Assume $d \geq 1$. Then the column vector $(\alpha''_0, \alpha''_1, \dots, \alpha''_d)^t$ is an eigenvector for the matrix*

$$\begin{pmatrix} 0 & \varphi_1/\varphi''_1 & & & & \mathbf{0} \\ -1/\varphi'_d & 0 & \varphi_2/\varphi''_1 & & & \\ & -1/\varphi'_d & 0 & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \varphi_d/\varphi''_1 \\ \mathbf{0} & & & & -1/\varphi'_d & 0 \end{pmatrix}. \quad (71)$$

The corresponding eigenvalue is α_1 .

Proof. In Lemma 13.58, represent everything with respect to $\{B^i\eta\}_{i=0}^d$, which is an inverted (B, A) -basis of V . In this calculation use Proposition 13.39 and the equation on the left in (68). \square

Lemma 13.60. *The column vector $(\alpha''_0, \alpha''_1, \dots, \alpha''_d)^t$ is an eigenvector for the matrix*

$$\begin{pmatrix} a_0 & \varphi'_d & & & & \mathbf{0} \\ \varphi''_d/\varphi_1 & a_1 & \varphi'_{d-1} & & & \\ & \varphi''_{d-1}/\varphi_2 & a_2 & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \varphi'_1 \\ \mathbf{0} & & & & \varphi''_1/\varphi_d & a_d \end{pmatrix}. \quad (72)$$

The corresponding eigenvalue is 0.

Proof. In the equation $C\eta'' = 0$, represent everything with respect to $\{B^i\eta\}_{i=0}^d$, which is an inverted (B, A) -basis of V . In this calculation use Proposition 13.39 and the equation on the left in (68). \square

Lemma 13.61. *For $0 \leq i \leq d$,*

$$\begin{aligned} \varphi_1\varphi_2\cdots\varphi_i\alpha''_i\beta'_d &= \beta'_{d-i}, & \varphi'_1\varphi'_2\cdots\varphi'_i\alpha_i\beta''_d &= \beta''_{d-i}, & \varphi''_1\varphi''_2\cdots\varphi''_i\alpha'_i\beta_d &= \beta_{d-i}, \\ \varphi_1\varphi_2\cdots\varphi_i\alpha'_i\beta''_d &= \beta''_{d-i}, & \varphi'_1\varphi'_2\cdots\varphi'_i\alpha''_i\beta_d &= \beta_{d-i}, & \varphi''_1\varphi''_2\cdots\varphi''_i\alpha_i\beta'_d &= \beta'_{d-i}. \end{aligned}$$

Proof. We verify the first equation. By Lemma 13.56 with $j = d$,

$$C^d\eta = \sum_{i=0}^d \beta'_{d-i} \frac{\varphi''_1\cdots\varphi''_d}{\varphi_1\cdots\varphi_i} B^i\eta. \quad (73)$$

The vectors $C^d\eta$ and η'' are both bases for C^dV , so there exists $0 \neq \vartheta \in \mathbb{F}$ such that $C^d\eta = \vartheta\eta''$. Use this to compare (73) with the equation on left in (68). We find that for $0 \leq i \leq d$,

$$\beta'_{d-i} \frac{\varphi''_1 \cdots \varphi''_d}{\varphi_1 \cdots \varphi_i} = \frac{(\eta'', \tilde{\eta}')}{(\eta, \tilde{\eta}')} \vartheta \alpha''_i. \quad (74)$$

Setting $i = 0$ in (74),

$$\beta'_d \varphi''_1 \cdots \varphi''_d = \frac{(\eta'', \tilde{\eta}')}{(\eta, \tilde{\eta}')} \vartheta. \quad (75)$$

Eliminating ϑ in (74) using (75), we obtain the first equation in the lemma statement. Apply this equation to the p-relatives of A, B, C to get the remaining equations in the lemma statement. \square

Lemma 13.62. *The following scalars are nonzero:*

$$\alpha_d, \quad \alpha'_d, \quad \alpha''_d, \quad \beta_d, \quad \beta'_d, \quad \beta''_d.$$

Moreover

$$\begin{aligned} \varphi_1 \varphi_2 \cdots \varphi_d &= \frac{1}{\alpha'_d \beta''_d} = \frac{1}{\alpha''_d \beta'_d}, & \varphi'_1 \varphi'_2 \cdots \varphi'_d &= \frac{1}{\alpha''_d \beta_d} = \frac{1}{\alpha_d \beta''_d}, \\ \varphi''_1 \varphi''_2 \cdots \varphi''_d &= \frac{1}{\alpha_d \beta'_d} = \frac{1}{\alpha'_d \beta_d}. \end{aligned}$$

Proof. Set $i = d$ in Lemma 13.61 and use Lemma 13.46. \square

Lemma 13.63. *For $0 \leq i \leq d$,*

$$\frac{\alpha_i}{\varphi_1 \varphi_2 \cdots \varphi_i} = \frac{\alpha'_i}{\varphi'_1 \varphi'_2 \cdots \varphi'_i} = \frac{\alpha''_i}{\varphi''_1 \varphi''_2 \cdots \varphi''_i} \quad (76)$$

and also

$$\frac{\beta_i}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-i+1}} = \frac{\beta'_i}{\varphi'_d \varphi'_{d-1} \cdots \varphi'_{d-i+1}} = \frac{\beta''_i}{\varphi''_d \varphi''_{d-1} \cdots \varphi''_{d-i+1}}. \quad (77)$$

Proof. To obtain (76), use Lemma 13.61 and the first assertion of Lemma 13.62. To obtain (77), apply (76) to the LR triple $\tilde{A}, \tilde{B}, \tilde{C}$ and use Lemmas 13.13, 13.51. \square

Lemma 13.64. *For $0 \leq j \leq d$,*

$$\begin{aligned} B^j \eta &= \frac{1}{\beta''_d} \frac{(\eta, \tilde{\eta}'')}{(\eta', \tilde{\eta}'')} \frac{A^{d-j} \eta'}{\varphi_d \cdots \varphi_{j+1}}, & C^j \eta &= \frac{1}{\alpha_d} \frac{(\eta, \tilde{\eta}')} {(\eta'', \tilde{\eta}')} \frac{A^{d-j} \eta''}{\varphi''_1 \cdots \varphi''_{d-j}}, \\ C^j \eta' &= \frac{1}{\beta_d} \frac{(\eta', \tilde{\eta})}{(\eta'', \tilde{\eta})} \frac{B^{d-j} \eta''}{\varphi'_d \cdots \varphi'_{j+1}}, & A^j \eta' &= \frac{1}{\alpha'_d} \frac{(\eta', \tilde{\eta}'')}{(\eta, \tilde{\eta}'')} \frac{B^{d-j} \eta}{\varphi_1 \cdots \varphi_{d-j}}, \\ A^j \eta'' &= \frac{1}{\beta'_d} \frac{(\eta'', \tilde{\eta}')} {(\eta, \tilde{\eta}')} \frac{C^{d-j} \eta'}{\varphi''_d \cdots \varphi''_{j+1}}, & B^j \eta'' &= \frac{1}{\alpha''_d} \frac{(\eta'', \tilde{\eta})}{(\eta', \tilde{\eta})} \frac{C^{d-j} \eta'}{\varphi'_1 \cdots \varphi'_{d-j}}. \end{aligned}$$

Proof. We verify the first equation. In Lemma 13.53, compare the (A, B) -basis of V from the first table, with the (A, B) -basis of V from the second table. By Lemma 3.22 there exists $0 \neq \zeta \in \mathbb{F}$ such that

$$\frac{B^j \eta}{\varphi_1 \cdots \varphi_j} = \zeta A^{d-j} \eta' \quad (0 \leq j \leq d). \quad (78)$$

We now find ζ . Setting $j = 0$ in (78) we find $\eta = \zeta A^d \eta'$. Use this to compare the equation in Lemma 13.56 (row 2, column 2, $j = d$), with the equation on the left in (66). In this comparison consider the summands for $i = 0$ to obtain

$$\zeta = \frac{1}{\beta_d''} \frac{1}{\varphi_1 \cdots \varphi_d} \frac{(\eta, \tilde{\eta}'')}{(\eta', \tilde{\eta}'')}. \quad (79)$$

Evaluating (78) using (79) we get the first displayed equation in the lemma statement. Applying our results so far to the LR triples in Lemma 13.9, we obtain the remaining equations in the lemma statement. \square

We emphasize a special case of Lemma 13.64.

Lemma 13.65. *We have*

$$\begin{aligned} B^d \eta &= \frac{(\eta, \tilde{\eta}'')}{(\eta', \tilde{\eta}'')} \frac{\eta'}{\beta_d''}, & C^d \eta &= \frac{(\eta, \tilde{\eta}')}{(\eta'', \tilde{\eta}')} \frac{\eta''}{\alpha_d}, \\ C^d \eta' &= \frac{(\eta', \tilde{\eta})}{(\eta'', \tilde{\eta})} \frac{\eta''}{\beta_d}, & A^d \eta' &= \frac{(\eta', \tilde{\eta}'')}{(\eta, \tilde{\eta}'')} \frac{\eta}{\alpha_d'}, \\ A^d \eta'' &= \frac{(\eta'', \tilde{\eta}')}{(\eta, \tilde{\eta}')} \frac{\eta}{\beta_d'}, & B^d \eta'' &= \frac{(\eta'', \tilde{\eta})}{(\eta', \tilde{\eta})} \frac{\eta'}{\alpha_d''}. \end{aligned}$$

Proof. Set $j = d$ in Lemma 13.64. \square

Proposition 13.66. *Each of α_d/β_d , α_d'/β_d' , α_d''/β_d'' is equal to*

$$\frac{(\eta, \tilde{\eta}')(\eta', \tilde{\eta}'')(\eta'', \tilde{\eta})}{(\eta, \tilde{\eta}'')(\eta', \tilde{\eta})(\eta'', \tilde{\eta}')}. \quad (80)$$

Proof. The scalars α_d/β_d , α_d'/β_d' , α_d''/β_d'' are equal by Lemma 13.62. To see that α_d/β_d is equal to (80), compare the two equations in (66) using Lemma 13.64 (row 2, column 1). The result follows after a brief computation. \square

Note 13.67. By Proposition 13.66 the scalars in (63), (64) are determined by the Toeplitz data (54) and the sequence

$$(\eta, \tilde{\eta}'), \quad (\eta', \tilde{\eta}''), \quad (\eta'', \tilde{\eta}), \quad (\eta, \tilde{\eta}''), \quad (\eta', \tilde{\eta}). \quad (81)$$

The scalars (81) are “free” in the following sense. Given a sequence Ψ of five nonzero scalars in \mathbb{F} , there exist nonzero vectors η, η', η'' and $\tilde{\eta}, \tilde{\eta}', \tilde{\eta}''$ as in (59), (60) such that the sequence (81) is equal to Ψ .

We display some transition matrices for later use.

Lemma 13.68. *Referring to Lemma 13.53, the following (i)–(iii) hold.*

- (i) *The transition matrix from the inverted (B, A) -basis in the first table to the inverted (B, A) -basis in the second table is*

$$\frac{(\eta', \tilde{\eta}'')}{(\eta, \tilde{\eta}'')} \beta_d'' I.$$

- (ii) *The transition matrix from the inverted (C, B) -basis in the second table to the inverted (C, B) -basis in the third table is*

$$\frac{(\eta'', \tilde{\eta})}{(\eta', \tilde{\eta})} \beta_d I.$$

- (iii) *The transition matrix from the inverted (A, C) -basis in the third table to the inverted (A, C) -basis in the first table is*

$$\frac{(\eta, \tilde{\eta}')}{(\eta'', \tilde{\eta}')} \beta_d' I.$$

Proof. Use Lemma 13.64. □

The following definition is motivated by Definition 3.47.

Definition 13.69. Let D (resp. D') (resp. D'') denote the diagonal matrix in $\text{Mat}_{d+1}(\mathbb{F})$ with (i, i) -entry $\varphi_1 \varphi_2 \cdots \varphi_i$ (resp. $\varphi'_1 \varphi'_2 \cdots \varphi'_i$) (resp. $\varphi''_1 \varphi''_2 \cdots \varphi''_i$) for $0 \leq i \leq d$.

The following result is reminiscent of Lemma 3.48.

Lemma 13.70. *Referring to Lemma 13.53,*

- (i) *D is the transition matrix from the (A, B) -basis in the first table to the inverted (B, A) -basis in the first table;*
(ii) *D' is the transition matrix from the (B, C) -basis in the second table to the inverted (C, B) -basis in the second table;*
(iii) *D'' is the transition matrix from the (C, A) -basis in the third table to the inverted (A, C) -basis in the third table.*

Proof. By Lemma 13.53 and Definition 13.69. □

Definition 13.71. Let θ denote the scalar (80). By Proposition 13.66,

$$\theta = \frac{\alpha_d}{\beta_d} = \frac{\alpha'_d}{\beta'_d} = \frac{\alpha''_d}{\beta''_d}. \quad (82)$$

Note that $\theta \neq 0$.

Proposition 13.72. *We have*

$$T'DZT''D'ZTD''Z = \frac{I}{\theta\beta_d\beta'_d\beta''_d}, \quad (83)$$

where θ is from Definition 13.71.

Proof. Consider the following 12 bases from Lemma 13.53. In each row of the table below, for $0 \leq i \leq d$ we display a vector $v_i \in V$. The vectors $\{v_i\}_{i=0}^d$ form a basis for V ; we give the type and the induced decomposition of V .

v_i	type of basis	induced dec. of V
$(\varphi_1 \cdots \varphi_i)^{-1} B^i \eta$	(A, B)	(A, B)
$B^i \eta$	inverted (B, A)	(A, B)
$(\varphi_d \cdots \varphi_{i+1})^{-1} A^{d-i} \eta'$	inverted (B, A)	(A, B)
$(\varphi_d \cdots \varphi_{d-i+1})^{-1} A^i \eta'$	(B, A)	(B, A)
$(\varphi'_1 \cdots \varphi'_i)^{-1} C^i \eta'$	(B, C)	(B, C)
$C^i \eta'$	inverted (C, B)	(B, C)
$(\varphi'_d \cdots \varphi'_{i+1})^{-1} B^{d-i} \eta''$	inverted (C, B)	(B, C)
$(\varphi'_d \cdots \varphi'_{d-i+1})^{-1} B^i \eta''$	(C, B)	(C, B)
$(\varphi''_1 \cdots \varphi''_i)^{-1} A^i \eta''$	(C, A)	(C, A)
$A^i \eta''$	inverted (A, C)	(C, A)
$(\varphi''_d \cdots \varphi''_{i+1})^{-1} C^{d-i} \eta$	inverted (A, C)	(C, A)
$(\varphi''_d \cdots \varphi''_{d-i+1})^{-1} C^i \eta$	(A, C)	(A, C)

We cycle through the bases in the above table, starting with the basis in the bottom row, jumping to the basis in the top row, and then going down through the rows until we return to the basis in the bottom row. For each basis in the sequence, consider the transition matrix to the next basis in the sequence. This gives a sequence of transition matrices. Compute the product of these transition matrices in the given order. This product is evaluated in two ways. On one hand, the product is equal to the identity matrix. On the other hand, each factor in the product is computed using Lemmas 13.55, 13.68, 13.70 and the definition of \mathbf{Z} in Section 2. Evaluate the resulting equation using Proposition 13.66. The result follows. \square

In Section 34 we use the equation (83) to characterize the LR triples.

We mention some other results involving the scalar θ from Definition 13.71.

Lemma 13.73. *We have*

$$\text{tr}(A^d B^d C^d) = \frac{\theta}{\alpha_d \alpha'_d \alpha''_d}, \quad \text{tr}(C^d B^d A^d) = \frac{1}{\theta \beta_d \beta'_d \beta''_d}. \quad (84)$$

Proof. We verify the equation on the left in (84). Let $\{u_i\}_{i=0}^d$ denote an (A, C) -basis of V , such that $u_0 = \eta$. Let S denote the matrix in $\text{Mat}_{d+1}(\mathbb{F})$ that represents $A^d B^d C^d$ with respect to $\{u_i\}_{i=0}^d$. The map C raises the (A, C) -decomposition of V . Therefore $C^d u_i = 0$ for $1 \leq i \leq d$. By Lemma 13.65 and Definition 13.71,

$$A^d B^d C^d \eta = \frac{\theta \eta}{\alpha_d \alpha'_d \alpha''_d}.$$

By these comments S has $(0, 0)$ -entry $\theta/(\alpha_d \alpha'_d \alpha''_d)$ and all other entries zero. We have verified the equation on the left in (84). The other equation is similarly verified. \square

Lemma 13.74. *For $0 \leq i \leq d$, the trace of $E'_{d-i} E_i E''_{d-i} E'_i E_{d-i} E''_i$ is*

$$\theta \frac{\varphi_d \cdots \varphi_{d-i+1}}{\varphi_1 \cdots \varphi_i} \frac{\varphi'_d \cdots \varphi'_{d-i+1}}{\varphi'_1 \cdots \varphi'_i} \frac{\varphi''_d \cdots \varphi''_{d-i+1}}{\varphi''_1 \cdots \varphi''_i},$$

and the trace of $E_{d-i} E'_i E''_{d-i} E_i E'_{d-i} E''_i$ is

$$\frac{1}{\theta} \frac{\varphi_1 \cdots \varphi_i}{\varphi_d \cdots \varphi_{d-i+1}} \frac{\varphi'_1 \cdots \varphi'_i}{\varphi'_d \cdots \varphi'_{d-i+1}} \frac{\varphi''_1 \cdots \varphi''_i}{\varphi''_d \cdots \varphi''_{d-i+1}}.$$

Proof. We verify the first assertion. In the product $E'_{d-i} E_i E''_{d-i} E'_i E_{d-i} E''_i$, evaluate each factor using Lemma 13.25, and simplify the result using Lemma 13.73 along with the meaning of the parameter array. The first assertion follows after a brief computation. The second assertion is similarly verified. \square

Corollary 13.75. *The trace of $E'_d E_0 E''_d E'_0 E_d E''_0$ is θ . The trace of $E_d E'_0 E''_d E_0 E'_d E''_0$ is θ^{-1} .*

Proof. Set $i = 0$ in Lemma 13.74. \square

14 How the parameter array, trace data, and Toeplitz data are related, I

Throughout this section and the next, let V denote a vector space over \mathbb{F} with dimension $d+1$. Fix an LR triple A, B, C on V . We consider how its parameter array (44), trace data (50), and Toeplitz data (54) are related.

Recall Definition 13.38. Let $\{u_i\}_{i=0}^d$ denote a basis for V of type $\natural = (A, C)$. Let $C^\natural \in \text{Mat}_{d+1}(\mathbb{F})$ represent C with respect to $\{u_i\}_{i=0}^d$. The entries of C^\natural are given in Proposition 13.39, row (A, C) of the table. Let $\{v_i\}_{i=0}^d$ denote a compatible basis for V of type $\sharp = (A, B)$. Let $C^\sharp \in \text{Mat}_{d+1}(\mathbb{F})$ represent C with respect to $\{v_i\}_{i=0}^d$. The entries of C^\sharp are given in Proposition 13.39, row (A, B) of the table. By Definition 13.44(ii), T' is the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$. By linear algebra,

$$C^\natural T' = T' C^\sharp. \quad (85)$$

Consequently

$$(T')^{-1} C^\natural T' = C^\sharp. \quad (86)$$

Proposition 14.1. *For $0 \leq i \leq d$,*

$$a_{d-i} = \alpha'_0 \beta'_1 \varphi''_i + \alpha'_1 \beta'_0 \varphi''_{i+1} = \alpha''_0 \beta''_1 \varphi'_i + \alpha''_1 \beta''_0 \varphi'_{i+1}, \quad (87)$$

$$a'_{d-i} = \alpha''_0 \beta''_1 \varphi_i + \alpha''_1 \beta''_0 \varphi_{i+1} = \alpha_0 \beta_1 \varphi''_i + \alpha_1 \beta_0 \varphi''_{i+1}, \quad (88)$$

$$a''_{d-i} = \alpha_0 \beta_1 \varphi'_i + \alpha_1 \beta_0 \varphi'_{i+1} = \alpha'_0 \beta'_1 \varphi_i + \alpha'_1 \beta'_0 \varphi_{i+1}. \quad (89)$$

Proof. We verify the equation on the left in (87). In the equation (86), compute the $(d - i, d - i)$ -entry of each side, and evaluate the result using Proposition 13.39 and Definition 13.44. This yields the equation on the left in (87). To finish the proof, apply this equation to the relatives of A, B, C . \square

We mention some variations on Proposition 14.1.

Corollary 14.2. *For $0 \leq i \leq d$,*

$$\begin{aligned} a_0 + a_1 + \cdots + a_{d-i} &= \beta'_1 \varphi''_i = \beta''_1 \varphi'_i, \\ a'_0 + a'_1 + \cdots + a'_{d-i} &= \beta''_1 \varphi_i = \beta_1 \varphi''_i, \\ a''_0 + a''_1 + \cdots + a''_{d-i} &= \beta_1 \varphi'_i = \beta'_1 \varphi_i. \end{aligned}$$

Proof. To verify each equation, evaluate the sum on the left using Proposition 14.1, and simplify the result using Lemma 13.46. \square

Corollary 14.3. *For $0 \leq i \leq d$,*

$$\begin{aligned} a_d + a_{d-1} + \cdots + a_{d-i} &= \alpha'_1 \varphi''_{i+1} = \alpha''_1 \varphi'_{i+1}, \\ a'_d + a'_{d-1} + \cdots + a'_{d-i} &= \alpha''_1 \varphi_{i+1} = \alpha_1 \varphi''_{i+1}, \\ a''_d + a''_{d-1} + \cdots + a''_{d-i} &= \alpha_1 \varphi'_{i+1} = \alpha'_1 \varphi_{i+1}. \end{aligned}$$

Proof. To verify each equation, evaluate the sum on the left using Proposition 14.1, and simplify the result using Lemma 13.46. \square

Corollary 14.4. *We have*

$$\begin{aligned} a_0 &= \beta'_1 \varphi''_d = \beta''_1 \varphi'_d, & a'_0 &= \beta''_1 \varphi_d = \beta_1 \varphi''_d, & a''_0 &= \beta_1 \varphi'_d = \beta'_1 \varphi_d, \\ a_d &= \alpha'_1 \varphi''_1 = \alpha''_1 \varphi'_1, & a'_d &= \alpha''_1 \varphi_1 = \alpha_1 \varphi''_1, & a''_d &= \alpha_1 \varphi'_1 = \alpha'_1 \varphi_1. \end{aligned}$$

Proof. Set $i = d$ in Corollary 14.2, and $i = 0$ in Corollary 14.3. \square

Corollary 14.5. *For $1 \leq i \leq d$,*

$$\frac{\alpha_1}{\varphi_i} = \frac{\alpha'_1}{\varphi'_i} = \frac{\alpha''_1}{\varphi''_i}, \quad \frac{\beta_1}{\varphi_i} = \frac{\beta'_1}{\varphi'_i} = \frac{\beta''_1}{\varphi''_i}. \quad (90)$$

Proof. Use Corollaries 14.2, 14.3. \square

Proposition 14.6. *For $1 \leq i \leq d$,*

$$\begin{aligned} \frac{\varphi'_i}{\varphi''_{d-i+1}} &= \alpha'_0 \beta'_2 \varphi_{i-1} + \alpha'_1 \beta'_1 \varphi_i + \alpha'_2 \beta'_0 \varphi_{i+1}, \\ \frac{\varphi''_i}{\varphi_{d-i+1}} &= \alpha''_0 \beta''_2 \varphi'_{i-1} + \alpha''_1 \beta''_1 \varphi'_i + \alpha''_2 \beta''_0 \varphi'_{i+1}, \\ \frac{\varphi_i}{\varphi'_{d-i+1}} &= \alpha_0 \beta_2 \varphi''_{i-1} + \alpha_1 \beta_1 \varphi''_i + \alpha_2 \beta_0 \varphi''_{i+1} \end{aligned}$$

and also

$$\begin{aligned}\frac{\varphi_i''}{\varphi_{d-i+1}'} &= \alpha_0''\beta_2''\varphi_{i-1} + \alpha_1''\beta_1''\varphi_i + \alpha_2''\beta_0''\varphi_{i+1}, \\ \frac{\varphi_i}{\varphi_{d-i+1}''} &= \alpha_0\beta_2\varphi_{i-1}' + \alpha_1\beta_1\varphi_i' + \alpha_2\beta_0\varphi_{i+1}', \\ \frac{\varphi_i'}{\varphi_{d-i+1}} &= \alpha_0'\beta_2'\varphi_{i-1}'' + \alpha_1'\beta_1'\varphi_i'' + \alpha_2'\beta_0'\varphi_{i+1}''.\end{aligned}$$

Proof. We verify the last equation in the proposition statement. In the equation (86), compute the $(d-i, d-i+1)$ -entry of each side, and evaluate the result using Proposition 13.39 and Definition 13.44. This yields the last equation in the proposition statement. To finish the proof, apply this equation to the relatives of A, B, C . \square

Proposition 14.7. For $3 \leq r \leq d+1$ and $0 \leq i \leq d-r+1$,

$$\begin{aligned}0 &= \alpha_0'\beta_r'\varphi_i + \alpha_1'\beta_{r-1}'\varphi_{i+1} + \cdots + \alpha_r'\beta_0'\varphi_{i+r}, \\ 0 &= \alpha_0''\beta_r''\varphi_i' + \alpha_1''\beta_{r-1}''\varphi_{i+1}' + \cdots + \alpha_r''\beta_0''\varphi_{i+r}', \\ 0 &= \alpha_0\beta_r\varphi_i'' + \alpha_1\beta_{r-1}\varphi_{i+1}'' + \cdots + \alpha_r\beta_0\varphi_{i+r}''\end{aligned}$$

and also

$$\begin{aligned}0 &= \alpha_0''\beta_r''\varphi_i + \alpha_1''\beta_{r-1}''\varphi_{i+1} + \cdots + \alpha_r''\beta_0''\varphi_{i+r}, \\ 0 &= \alpha_0\beta_r\varphi_i' + \alpha_1\beta_{r-1}\varphi_{i+1}' + \cdots + \alpha_r\beta_0\varphi_{i+r}', \\ 0 &= \alpha_0'\beta_r'\varphi_i'' + \alpha_1'\beta_{r-1}'\varphi_{i+1}'' + \cdots + \alpha_r'\beta_0'\varphi_{i+r}''.\end{aligned}$$

Proof. We verify the last equation in the proposition statement. In the equation (86), compute the $(d-i-r+1, d-i)$ -entry of each side, and evaluate the result using Proposition 13.39 and Definition 13.44. This yields the last equation in the proposition statement. To finish the proof, apply this equation to the relatives of A, B, C . \square

15 How the parameter array, trace data, and Toeplitz data are related, II

We continue to discuss our LR triple A, B, C on V , with parameter array (44), trace data (50), and Toeplitz data (54). In the previous section we found a relationship among these scalars, using the equation (86). In the present section we describe this relationship from the point of view of (85).

Proposition 15.1. For $1 \leq i \leq d$ and $0 \leq j \leq d-i$,

$$\begin{aligned}\alpha_{i-1}'\frac{\varphi_{j+1}'}{\varphi_{d-j}''} + \alpha_i'a_{d-j}'' + \alpha_{i+1}'\varphi_j &= \alpha_{i+1}'\varphi_{i+j+1}, \\ \alpha_{i-1}''\frac{\varphi_{j+1}'}{\varphi_{d-j}''} + \alpha_i''a_{d-j}'' + \alpha_{i+1}''\varphi_j' &= \alpha_{i+1}''\varphi_{i+j+1}', \\ \alpha_{i-1}\frac{\varphi_{j+1}'}{\varphi_{d-j}'} + \alpha_i a_{d-j}' + \alpha_{i+1}\varphi_j'' &= \alpha_{i+1}\varphi_{i+j+1}''\end{aligned}$$

and also

$$\begin{aligned}\alpha''_{i-1} \frac{\varphi''_{j+1}}{\varphi'_{d-j}} + \alpha''_i a'_{d-j} + \alpha''_{i+1} \varphi_j &= \alpha''_{i+1} \varphi_{i+j+1}, \\ \alpha_{i-1} \frac{\varphi_{j+1}}{\varphi''_{d-j}} + \alpha_i a''_{d-j} + \alpha_{i+1} \varphi'_j &= \alpha_{i+1} \varphi'_{i+j+1}, \\ \alpha'_{i-1} \frac{\varphi'_{j+1}}{\varphi'_{d-j}} + \alpha'_i a_{d-j} + \alpha'_{i+1} \varphi''_j &= \alpha'_{i+1} \varphi''_{i+j+1}.\end{aligned}$$

Proof. We verify the last equation in the proposition statement. In the equation (85), compute the $(d-i-j, d-j)$ -entry of each side, and evaluate the result using Proposition 13.39 and Definition 13.44. This yields the last equation of the proposition statement. To finish the proof, apply this equation to the p -relatives of A, B, C . \square

We point out some special cases of Proposition 15.1.

Corollary 15.2. *For $1 \leq i \leq d-1$,*

$$\begin{aligned}\alpha'_{i-1} \frac{\varphi'_{d-i+1}}{\varphi''_i} + \alpha'_i a''_i + \alpha'_{i+1} \varphi_{d-i} &= 0, \\ \alpha''_{i-1} \frac{\varphi''_{d-i+1}}{\varphi_i} + \alpha''_i a_i + \alpha''_{i+1} \varphi'_{d-i} &= 0, \\ \alpha_{i-1} \frac{\varphi_{d-i+1}}{\varphi'_i} + \alpha_i a'_i + \alpha_{i+1} \varphi''_{d-i} &= 0\end{aligned}$$

and also

$$\begin{aligned}\alpha''_{i-1} \frac{\varphi''_{d-i+1}}{\varphi'_i} + \alpha''_i a'_i + \alpha''_{i+1} \varphi_{d-i} &= 0, \\ \alpha_{i-1} \frac{\varphi_{d-i+1}}{\varphi''_i} + \alpha_i a''_i + \alpha_{i+1} \varphi'_{d-i} &= 0, \\ \alpha'_{i-1} \frac{\varphi'_{d-i+1}}{\varphi_i} + \alpha'_i a_i + \alpha'_{i+1} \varphi''_{d-i} &= 0.\end{aligned}$$

Proof. In Proposition 15.1, assume $i \leq d-1$ and $j = d-i$. \square

Corollary 15.3. *For $1 \leq i \leq d-1$,*

$$\begin{aligned}\alpha'_{i-1} \frac{\varphi'_1}{\varphi''_d} + \alpha'_i a''_d &= \alpha'_{i+1} \varphi_{i+1}, & \alpha''_{i-1} \frac{\varphi''_1}{\varphi'_d} + \alpha''_i a'_d &= \alpha''_{i+1} \varphi'_{i+1}, \\ \alpha''_{i-1} \frac{\varphi''_1}{\varphi_d} + \alpha''_i a_d &= \alpha''_{i+1} \varphi'_{i+1}, & \alpha_{i-1} \frac{\varphi_1}{\varphi''_d} + \alpha_i a''_d &= \alpha_{i+1} \varphi'_{i+1}, \\ \alpha_{i-1} \frac{\varphi_1}{\varphi'_d} + \alpha_i a'_d &= \alpha_{i+1} \varphi''_{i+1}, & \alpha'_{i-1} \frac{\varphi'_1}{\varphi_d} + \alpha'_i a_d &= \alpha'_{i+1} \varphi''_{i+1}.\end{aligned}$$

Proof. In Proposition 15.1, assume $i \leq d-1$ and $j = 0$. \square

Corollary 15.4. For $d \geq 1$,

$$\begin{aligned}\alpha'_{d-1} \frac{\varphi'_1}{\varphi''_d} + \alpha'_d a''_d &= 0, & \alpha''_{d-1} \frac{\varphi''_1}{\varphi_d} + \alpha''_d a_d &= 0, & \alpha_{d-1} \frac{\varphi_1}{\varphi'_d} + \alpha_d a'_d &= 0, \\ \alpha''_{d-1} \frac{\varphi''_1}{\varphi'_d} + \alpha''_d a'_d &= 0, & \alpha_{d-1} \frac{\varphi_1}{\varphi''_d} + \alpha_d a''_d &= 0, & \alpha'_{d-1} \frac{\varphi'_1}{\varphi_d} + \alpha'_d a_d &= 0.\end{aligned}$$

Proof. In Proposition 15.1, assume $i = d$ and $j = 0$. □

Proposition 15.5. For $1 \leq i \leq d$ and $0 \leq j \leq d - i$,

$$\begin{aligned}\beta'_{i-1} \frac{\varphi'_{d-j}}{\varphi''_{j+1}} + \beta'_i a''_j + \beta'_{i+1} \varphi_{d-j+1} &= \beta'_{i+1} \varphi_{d-i-j}, \\ \beta''_{i-1} \frac{\varphi''_{d-j}}{\varphi_{j+1}} + \beta''_i a_j + \beta''_{i+1} \varphi'_{d-j+1} &= \beta''_{i+1} \varphi'_{d-i-j}, \\ \beta_{i-1} \frac{\varphi_{d-j}}{\varphi'_{j+1}} + \beta_i a'_j + \beta_{i+1} \varphi''_{d-j+1} &= \beta_{i+1} \varphi''_{d-i-j}\end{aligned}$$

and also

$$\begin{aligned}\beta''_{i-1} \frac{\varphi''_{d-j}}{\varphi'_{j+1}} + \beta''_i a'_j + \beta''_{i+1} \varphi_{d-j+1} &= \beta''_{i+1} \varphi_{d-i-j}, \\ \beta_{i-1} \frac{\varphi_{d-j}}{\varphi''_{j+1}} + \beta_i a''_j + \beta_{i+1} \varphi'_{d-j+1} &= \beta_{i+1} \varphi'_{d-i-j}, \\ \beta'_{i-1} \frac{\varphi'_{d-j}}{\varphi_{j+1}} + \beta'_i a_j + \beta'_{i+1} \varphi''_{d-j+1} &= \beta'_{i+1} \varphi''_{d-i-j}.\end{aligned}$$

Proof. Apply Proposition 15.1 to the LR triple $\tilde{A}, \tilde{B}, \tilde{C}$. □

Corollary 15.6. For $1 \leq i \leq d - 1$,

$$\begin{aligned}\beta'_{i-1} \frac{\varphi'_i}{\varphi''_{d-i+1}} + \beta'_i a''_{d-i} + \beta'_{i+1} \varphi_{i+1} &= 0, \\ \beta''_{i-1} \frac{\varphi''_i}{\varphi_{d-i+1}} + \beta''_i a_{d-i} + \beta''_{i+1} \varphi'_{i+1} &= 0, \\ \beta_{i-1} \frac{\varphi_i}{\varphi'_{d-i+1}} + \beta_i a'_{d-i} + \beta_{i+1} \varphi''_{i+1} &= 0\end{aligned}$$

and also

$$\begin{aligned}\beta''_{i-1} \frac{\varphi''_i}{\varphi'_{d-i+1}} + \beta''_i a'_{d-i} + \beta''_{i+1} \varphi_{i+1} &= 0, \\ \beta_{i-1} \frac{\varphi_i}{\varphi''_{d-i+1}} + \beta_i a''_{d-i} + \beta_{i+1} \varphi'_{i+1} &= 0, \\ \beta'_{i-1} \frac{\varphi'_i}{\varphi_{d-i+1}} + \beta'_i a_{d-i} + \beta'_{i+1} \varphi''_{i+1} &= 0.\end{aligned}$$

Proof. In Proposition 15.5, assume $i \leq d - 1$ and $j = d - i$. □

Corollary 15.7. For $1 \leq i \leq d - 1$,

$$\begin{aligned} \beta'_{i-1} \frac{\varphi'_d}{\varphi''_1} + \beta'_i a''_0 &= \beta'_{i+1} \varphi_{d-i}, & \beta''_{i-1} \frac{\varphi''_d}{\varphi'_1} + \beta''_i a'_0 &= \beta''_{i+1} \varphi_{d-i}, \\ \beta''_{i-1} \frac{\varphi''_d}{\varphi'_1} + \beta''_i a'_0 &= \beta''_{i+1} \varphi'_{d-i}, & \beta_{i-1} \frac{\varphi_d}{\varphi''_1} + \beta_i a''_0 &= \beta_{i+1} \varphi'_{d-i}, \\ \beta_{i-1} \frac{\varphi_d}{\varphi''_1} + \beta_i a''_0 &= \beta_{i+1} \varphi''_{d-i}, & \beta'_{i-1} \frac{\varphi'_d}{\varphi_1} + \beta'_i a'_0 &= \beta'_{i+1} \varphi''_{d-i}. \end{aligned}$$

Proof. In Proposition 15.5, assume $i \leq d - 1$ and $j = 0$. □

Corollary 15.8. For $d \geq 1$,

$$\begin{aligned} \beta'_{d-1} \frac{\varphi'_d}{\varphi''_1} + \beta'_d a''_0 &= 0, & \beta''_{d-1} \frac{\varphi''_d}{\varphi'_1} + \beta''_d a'_0 &= 0, & \beta_{d-1} \frac{\varphi_d}{\varphi''_1} + \beta_d a''_0 &= 0, \\ \beta''_{d-1} \frac{\varphi''_d}{\varphi'_1} + \beta''_d a'_0 &= 0, & \beta_{d-1} \frac{\varphi_d}{\varphi''_1} + \beta_d a''_0 &= 0, & \beta'_{d-1} \frac{\varphi'_d}{\varphi_1} + \beta'_d a'_0 &= 0. \end{aligned}$$

Proof. In Proposition 15.5, assume $i = d$ and $j = 0$. □

We have displayed many equations relating the parameter array (44), trace data (50), and Toeplitz data (54). From these equations it is apparent that we can improve on Proposition 13.40. We now give some results in this direction. To avoid trivialities we assume $d \geq 1$.

Proposition 15.9. Assume $d \geq 1$. Then the LR triple A, B, C is uniquely determined up to isomorphism by its parameter array along with any one of the following 12 scalars:

$$a_0, a'_0, a''_0; \quad a_d, a'_d, a''_d; \quad \alpha_1, \alpha'_1, \alpha''_1; \quad \beta_1, \beta'_1, \beta''_1. \quad (91)$$

Proof. Use Proposition 13.40 along with (57), Proposition 14.1, and Corollary 14.4. □

In our discussion going forward, among the scalars (91) we will put the emphasis on α_1 . We call α_1 the *first Toeplitz number* of the LR triple A, B, C .

Lemma 15.10. Assume $d \geq 1$. For the LR triple A, B, C let \mathbb{K} denote a subfield of \mathbb{F} that contains the scalars (44) and the first Toeplitz number α_1 . Then there exists an LR triple over \mathbb{K} that has parameter array (44) and first Toeplitz number α_1 .

Proof. Represent A, B, C by matrices, using the first row in the table of Proposition 13.39. For the resulting three matrices each entry is in \mathbb{K} . So each matrix represents a \mathbb{K} -linear transformation of a vector space over \mathbb{K} . The resulting three \mathbb{K} -linear transformations form an LR triple over \mathbb{K} that has parameter array (44) and first Toeplitz number α_1 . □

Lemma 15.11. For the LR triple A, B, C the following are equivalent:

- (i) $\varphi_i = \varphi'_i = \varphi''_i$ for $1 \leq i \leq d$;
- (ii) the p -relatives of A, B, C are mutually isomorphic;

(iii) the n -relatives of A, B, C are mutually isomorphic.

Assume that (i)–(iii) hold. Then for $0 \leq i \leq d$,

$$a_i = a'_i = a''_i, \quad \alpha_i = \alpha'_i = \alpha''_i, \quad \beta_i = \beta'_i = \beta''_i. \quad (92)$$

Proof. Assume $d \geq 1$; otherwise (i)–(iii) and (92) all hold.

(i) \Rightarrow (ii) We have $\alpha_1 = \alpha'_1 = \alpha''_1$ by Corollary 14.5. The result follows by Proposition 15.9, along with Lemmas 13.9, 13.13 and Definition 13.14.

(ii) \Rightarrow (i) By Lemmas 13.9, 13.13 and Definition 13.14.

(i) \Leftrightarrow (iii) Similar to the proof of (i) \Leftrightarrow (ii) above.

Assume that (i)–(iii) hold. Then (92) holds by Lemmas 13.36, 13.37, 13.49, 13.51. \square

We now compute the Toeplitz data (54) in terms of the parameter array (44) and any scalar from (91). We will focus on $\{\alpha_i\}_{i=0}^d$ and $\{\beta_i\}_{i=0}^d$.

Proposition 15.12. *For $d \geq 1$ the following (i), (ii) hold.*

(i) *The sequence $\{\alpha_i\}_{i=0}^d$ is computed as follows: $\alpha_0 = 1$ and α_1 is from Corollary 14.4. Moreover*

$$\alpha_{i+1} = \frac{\alpha_1 \alpha_i \varphi''_1 + \alpha_{i-1} \varphi_1 (\varphi'_d)^{-1}}{\varphi''_{i+1}} \quad (1 \leq i \leq d-1). \quad (93)$$

(ii) *The sequence $\{\beta_i\}_{i=0}^d$ is computed as follows: $\beta_0 = 1$ and β_1 is from Corollary 14.4. Moreover*

$$\beta_{i+1} = \frac{\beta_1 \beta_i \varphi''_d + \beta_{i-1} \varphi_d (\varphi'_1)^{-1}}{\varphi''_{d-i}} \quad (1 \leq i \leq d-1). \quad (94)$$

Proof. (i) We verify (93). Consider the displayed equation in Corollary 15.3 (row 3, column 1). In this equation solve for α_{i+1} , and eliminate a'_d using the equation $a'_d = \alpha_1 \varphi''_1$ from Corollary 14.4.

(ii) Similar to the proof of (i) above. \square

We now give some more ways to compute $\{\alpha_i\}_{i=0}^d$ and $\{\beta_i\}_{i=0}^d$.

Proposition 15.13. *For $d \geq 1$ the following (i), (ii) hold.*

(i) *The sequence $\{\alpha_i\}_{i=0}^d$ is computed as follows: $\alpha_0 = 1$ and α_1 is from Corollary 14.4. Moreover*

$$\alpha_{i+1} = \frac{\alpha_1 \alpha_i (\varphi''_{d-i} - \varphi''_{d-i+1}) - \alpha_{i-1} \varphi_{d-i+1} (\varphi'_i)^{-1}}{\varphi''_{d-i}} \quad (1 \leq i \leq d-1). \quad (95)$$

(ii) *The sequence $\{\beta_i\}_{i=0}^d$ is computed as follows: $\beta_0 = 1$ and β_1 is from Corollary 14.4. Moreover*

$$\beta_{i+1} = \frac{\beta_1 \beta_i (\varphi''_{i+1} - \varphi''_i) - \beta_{i-1} \varphi_i (\varphi'_{d-i+1})^{-1}}{\varphi''_{i+1}} \quad (1 \leq i \leq d-1). \quad (96)$$

Proof. (i) We verify (95). Consider the displayed equation in Corollary 15.2 (row 3). In this equation solve for α_{i+1} , and eliminate a'_i using the equation $a'_i = \alpha_1(\varphi''_{d-i+1} - \varphi''_{d-i})$ from Proposition 14.1.

(ii) Similar to the proof of (i) above. \square

Proposition 15.14. *For $d \geq 1$ the following (i)–(iv) hold:*

(i) for $1 \leq i \leq d-1$,

$$\alpha_1 \alpha_i \left(1 - \frac{\varphi''_1}{\varphi''_{i+1}} - \frac{\varphi''_{d-i+1}}{\varphi''_{d-i}} \right) = \alpha_{i-1} \left(\frac{\varphi_1}{\varphi'_d \varphi''_{i+1}} + \frac{\varphi_{d-i+1}}{\varphi'_i \varphi''_{d-i}} \right);$$

(ii) $\alpha_1 \alpha_d \varphi''_1 = -\alpha_{d-1} \varphi_1 / \varphi'_d$;

(iii) for $1 \leq i \leq d-1$,

$$\beta_1 \beta_i \left(1 - \frac{\varphi''_d}{\varphi''_{d-i}} - \frac{\varphi''_i}{\varphi''_{i+1}} \right) = \beta_{i-1} \left(\frac{\varphi_d}{\varphi'_1 \varphi''_{d-i}} + \frac{\varphi_i}{\varphi'_{d-i+1} \varphi''_{i+1}} \right);$$

(iv) $\beta_1 \beta_d \varphi''_d = -\beta_{d-1} \varphi_d / \varphi'_1$.

Proof. (i) Subtract (95) from (93) and simplify the result.

(ii) In the displayed equation of Corollary 15.4 (row 1, column 3) eliminate a'_d using the equation $a'_d = \alpha_1 \varphi''_1$ from Corollary 14.4.

(iii), (iv) Similar to the proof of (i), (ii) above. \square

Note 15.15. Referring to Proposition 15.9, if we replace the LR triple A, B, C by the LR triple $-A, -B, -C$ then the parameter array is unchanged, and each scalar in (91) is replaced by its opposite. So in general, the LR triple A, B, C is not determined up to isomorphism by its parameter array.

Referring to Proposition 15.9 and in light of Note 15.15, we now consider the extent to which α_1^2 is determined by the parameter array (44).

Lemma 15.16. *For $d \geq 1$ the scalar α_1^2 is related to the parameter array (44) in the following way.*

(i) Assume $d = 1$. Then

$$\alpha_1^2 = -\frac{\varphi_1}{\varphi'_1 \varphi''_1}. \quad (97)$$

(ii) Assume $d = 2$. Then $\alpha_1 = 0$ or

$$\alpha_1^2 = -\frac{\varphi_1 + \varphi_2}{\varphi'_2 \varphi''_1}. \quad (98)$$

(iii) Assume $d \geq 2$. Then

$$\alpha_1^2 \left(1 - \frac{\varphi_1''}{\varphi_2''} - \frac{\varphi_d''}{\varphi_{d-1}''} \right) = \frac{\varphi_d}{\varphi_{d-1}''} \frac{1}{\varphi_1'} + \frac{\varphi_1}{\varphi_2''} \frac{1}{\varphi_d'}. \quad (99)$$

Moreover for $1 \leq i \leq d$,

$$\alpha_1^2 \left(\frac{\varphi_d''}{\varphi_{d-1}''} \varphi_{i-1}'' - \varphi_i'' + \frac{\varphi_1''}{\varphi_2''} \varphi_{i+1}'' \right) = \frac{\varphi_i}{\varphi_{d-i+1}'} - \frac{\varphi_d}{\varphi_{d-1}''} \frac{\varphi_{i-1}''}{\varphi_1'} - \frac{\varphi_1}{\varphi_2''} \frac{\varphi_{i+1}''}{\varphi_d'}. \quad (100)$$

Proof. (i), (ii) Compute the eigenvalues of the matrix (71).

(iii) Using Proposition 15.12, solve for α_2 , β_2 in terms of α_1 and the parameter array (44). To obtain (99), use the above solutions and $\beta_2 = \alpha_1^2 - \alpha_2$. To obtain (100), use the above solutions and the third displayed equation in Proposition 14.6. \square

Referring to Lemma 15.16(iii), it sometimes happens that in each equation (99), (100) the coefficient of α_1^2 is zero. We illustrate with two examples.

Definition 15.17. The LR triple A, B, C is said to have *Weyl type* whenever the LR pairs A, B and B, C and C, A all have Weyl type, in the sense of Definition 4.3. In this case, $p = d + 1$ is prime and $\text{Char}(\mathbb{F}) = p$. Moreover

$$AB - BA = I, \quad BC - CB = I, \quad CA - AC = I, \quad (101)$$

$$\varphi_i = \varphi_i' = \varphi_i'' = i \quad (1 \leq i \leq d). \quad (102)$$

Lemma 15.18. Assume that the LR triple A, B, C has Weyl type. Then each p -relative of A, B, C has Weyl type.

Proof. By Lemma 15.11 and (102). \square

Lemma 15.19. Assume that $d \geq 2$ and A, B, C has Weyl type. Then in each of (99), (100) the coefficient of α_1^2 is zero. Moreover the right-hand side is zero.

Proof. This is readily checked using (102). \square

Assume that A, B, C has Weyl type. Then equations (99), (100) give no information about α_1 . To compute α_1 we use the following result.

Lemma 15.20. Assume that A, B, C has Weyl type. Then

$$A + B + C = \alpha_1 I. \quad (103)$$

Proof. Represent A, B, C by matrices, using for example the first row in the table of Proposition 13.39. By Proposition 14.1 we have $a_i = \alpha_1$ for $0 \leq i \leq d$. \square

Lemma 15.21. Assume that A, B, C has Weyl type. Then $\alpha_1 = 1$ if $d = 1$, and $\alpha_1 = 0$ if $d \geq 2$.

Proof. Recall from Definition 15.17 that $p = d + 1$ is prime and $\text{Char}(\mathbb{F}) = p$. First assume that $d = 1$. Then by Lemma 15.16(i) and since $\text{Char}(\mathbb{F}) = 2$, we obtain $\alpha_1^2 = 1$. Again using $\text{Char}(\mathbb{F}) = 2$ we find $\alpha_1 = 1$. Next assume that $d \geq 2$. By Lemma 15.20, $C - \alpha_1 I = -A - B$. On one hand, the pair B, C is an LR pair on V , so C is Nil by Lemma 3.3. On the other hand, by Lemma 4.8 the pair $B, -A - B$ is an LR pair on V , so $-A - B$ is Nil by Lemma 3.3. By these comments, both C and $C - \alpha_1 I$ are Nil. Considering their eigenvalues we obtain $\alpha_1 = 0$. \square

Lemma 15.22. *Assume that $d \geq 2$ and A, B, C has Weyl type. Then*

$$\alpha_{2i} = \frac{(-1)^i}{2^i i!}, \quad \beta_{2i} = \frac{1}{2^i i!} \quad (0 \leq i \leq d/2).$$

Also $\alpha_{2i+1} = 0$ and $\beta_{2i+1} = 0$ for $0 \leq i < d/2$.

Proof. Use Proposition 15.12 and Lemma 15.21. \square

Proposition 15.23. *The following are equivalent:*

- (i) $p = d + 1$ is prime and $\text{Char}(\mathbb{F}) = p$;
- (ii) *there exists an LR triple A, B, C over \mathbb{F} that has diameter d and Weyl type.*

Assume that (i), (ii) hold. Then A, B, C is unique up to isomorphism.

Proof. (i) \Rightarrow (ii) By Lemma 4.6, there exists an LR pair A, B over \mathbb{F} that has diameter d and Weyl type. Define $C = I - A - B$ if $d = 1$, and $C = -A - B$ if $d \geq 2$. For $d = 1$ one routinely verifies that A, B, C is an LR triple of Weyl type. Assume that $d \geq 2$. By Lemma 4.8 and Definition 15.17 the sequence A, B, C is an LR triple of Weyl type.

(ii) \Rightarrow (i) By Definition 15.17.

Assume that (i), (ii) hold. The LR triple A, B, C is unique up to isomorphism by Proposition 15.9, line (102), and Lemma 15.21. \square

We continue to discuss our LR triple A, B, C on V , with parameter array (44), trace data (50), and Toeplitz data (54).

Definition 15.24. Pick a nonzero $q \in \mathbb{F}$ such that $q^2 \neq 1$. The LR triple A, B, C is said to have q -Weyl type whenever the LR pairs A, B and B, C and C, A all have q -Weyl type, in the sense of Definition 4.11. In this case d, q or $d, -q$ is standard. Moreover

$$\frac{qAB - q^{-1}BA}{q - q^{-1}} = I, \quad \frac{qBC - q^{-1}CB}{q - q^{-1}} = I, \quad \frac{qCA - q^{-1}AC}{q - q^{-1}} = I, \quad (104)$$

$$\varphi_i = \varphi'_i = \varphi''_i = 1 - q^{-2i} \quad (1 \leq i \leq d). \quad (105)$$

Lemma 15.25. *With reference to Definition 15.24, assume that A, B, C has q -Weyl type. Then A, B, C has $(-q)$ -Weyl type.*

Proof. Use Note 4.12. \square

Lemma 15.26. *With reference to Definition 15.24, assume that A, B, C has q -Weyl type. Then each p -relative of A, B, C has q -Weyl type. Moreover each n -relative of A, B, C has (q^{-1}) -Weyl type.*

Proof. By Note 4.12 and Definition 13.14. □

Lemma 15.27. *With reference to Definition 15.24, assume that $d \geq 2$ and A, B, C has q -Weyl type. Then in each of (99), (100) the coefficient of α_1^2 is zero. Moreover the right-hand side is zero.*

Proof. This is readily checked. □

With reference to Definition 15.24, assume that $d \geq 2$ and A, B, C has q -Weyl type. Then (99), (100) give no information about α_1 . To get some information about α_1 we turn to Lemma 13.58.

Lemma 15.28. *With reference to Definition 15.24, assume that A, B, C has q -Weyl type. Then*

$$A/\varphi_1 - B/\varphi_d = \frac{qA + q^{-1}B}{q - q^{-1}}. \quad (106)$$

Proof. Use (105). □

Recall Assumption 4.17.

Lemma 15.29. *With reference to Assumption 4.17 and Definition 15.24, assume that A, B, C has q -Weyl type. Then for the element (106) the roots of the characteristic polynomial are*

$$\frac{q^{j+1/2} + q^{-j-1/2}}{q - q^{-1}} \quad (0 \leq j \leq d). \quad (107)$$

Moreover α_1 is contained in the list (107).

Proof. The first assertion follows from Lemma 4.23. The second assertion follows from Lemma 13.58 and (105). □

We now consider which values of (107) could equal α_1 .

Lemma 15.30. *With reference to Definition 15.24, assume that A, B, C has q -Weyl type. Then $\alpha_1(q - q^{-1})I$ is equal to each of*

$$\begin{array}{ll} qA + q^{-1}B + qC - qABC, & q^{-1}A + qB + q^{-1}C - q^{-1}CBA, \\ qB + q^{-1}C + qA - qBCA, & q^{-1}B + qC + q^{-1}A - q^{-1}ACB, \\ qC + q^{-1}A + qB - qCAB, & q^{-1}C + qA + q^{-1}B - q^{-1}BAC. \end{array}$$

Proof. Represent A, B, C by matrices, using for example the first row in the table of Proposition 13.39. By Proposition 14.1 we have $a_i = \alpha_1 q^{2i+1}(q - q^{-1})$ for $0 \leq i \leq d$. □

Proposition 15.31. *Assume that \mathbb{F} is algebraically closed. With reference to Assumption 4.17, pick an integer j ($0 \leq j \leq d$) and define*

$$\alpha_1 = \frac{q^{j+1/2} + q^{-j-1/2}}{q - q^{-1}}. \quad (108)$$

Then there exists an LR triple over \mathbb{F} that has diameter d and q -Weyl type, with first Toeplitz number α_1 . This LR triple is uniquely determined up to isomorphism by d, q, j . For this LR triple,

$$\alpha_i = \frac{(-1)^i q^{-i/2}}{(q^{-1}; q^{-1})_i} {}_3\phi_2 \left(\begin{matrix} q^i, -q^{-j-1}, -q^j \\ 0, -q^{-1} \end{matrix} \middle| q^{-1}, q^{-1} \right) \quad (0 \leq i \leq d), \quad (109)$$

$$\beta_i = \frac{(-1)^i q^{i/2}}{(q; q)_i} {}_3\phi_2 \left(\begin{matrix} q^{-i}, -q^{j+1}, -q^{-j} \\ 0, -q \end{matrix} \middle| q, q \right) \quad (0 \leq i \leq d). \quad (110)$$

Proof. By Lemma 4.18 there exists an LR pair A, B on V that has q -Weyl type. Its parameter sequence $\{\varphi_i\}_{i=1}^d$ satisfies $\varphi_i = 1 - q^{-2i}$ for $1 \leq i \leq d$. With respect to an (A, B) -basis of V the matrices representing A, B are given as shown in the first row of the table in Proposition 13.39. Define $C \in \text{End}(V)$ such that with respect to the (A, B) -basis, the matrix representing C is given as shown in the first row of the table, using $a_i = \alpha_1 q^{2i+1}(q - q^{-1})$ for $0 \leq i \leq d$ and $\varphi'_i = \varphi''_i = \varphi_i$ for $1 \leq i \leq d$. We show that A, B, C is an LR triple on V that has q -Weyl type. To do this, it suffices to show that B, C and C, A are LR pairs on V that have q -Weyl type. We now show that B, C is an LR pair on V that has q -Weyl type. To this end we apply Lemma 4.19 to the pair B, C . From the matrix representations we see that B, C satisfy the middle equation in (104). The map B is not invertible, since B is Nil by Lemma 3.3. We show that C is not invertible. From the matrix representations we obtain

$$\alpha_1(q - q^{-1})I = qA + q^{-1}B + qC - qABC. \quad (111)$$

Rearranging (111),

$$\alpha_1(q - q^{-1})I - qA - q^{-1}B = q(1 - AB)C. \quad (112)$$

By assumption (108) along with Definition 4.21 and Lemma 4.23, $\alpha_1(q - q^{-1})$ is an eigenvalue for $qA + q^{-1}B$. So in the equation (112) the expression on the left is not invertible. Therefore $(1 - AB)C$ is not invertible. Note that $I - AB$ is diagonalizable with eigenvalues $\{1 - \varphi_{i+1}\}_{i=0}^d$. Moreover $1 - \varphi_{i+1} = q^{-2i-2} \neq 0$ for $0 \leq i \leq d$. Therefore $I - AB$ is invertible. By these comments C is not invertible. Applying Lemma 4.19 to the pair B, C we find that B, C is an LR pair on V that has q -Weyl type. One similarly shows that C, A is an LR pair on V that has q -Weyl type. Now by Definition 15.24 the triple A, B, C is an LR triple on V that has q -Weyl type. Comparing (111) with the first expression in the display of Lemma 15.30, we see that A, B, C has first Toeplitz number α_1 . We have displayed an LR triple over \mathbb{F} that has diameter d and q -Weyl type, with first Toeplitz number α_1 . This LR triple is unique up to isomorphism by Proposition 15.9 and line (105). To obtain (109) use the eigenvector assertion in Lemma 4.20 along with Lemma 13.59. To obtain (110), apply (109) to any n -relative of A, B, C and use Lemma 15.26. \square

16 Bipartite LR triples

Throughout this section the following notation is in effect. Let V denote a vector space over \mathbb{F} with dimension $d + 1$. Let A, B, C denote an LR triple on V , with parameter array (44), idempotent data (48), trace data (50), and Toeplitz data (54). We describe a condition on A, B, C called bipartite.

Definition 16.1. The LR triple A, B, C is called *bipartite* whenever each of a_i, a'_i, a''_i is zero for $0 \leq i \leq d$.

Lemma 16.2. *If A, B, C is trivial then it is bipartite.*

Proof. Set $d = 0$ in Lemma 13.34 to see that each of a_0, a'_0, a''_0 is zero. \square

Lemma 16.3. *Assume that A, B, C is bipartite (resp. nonbipartite). Then each relative of A, B, C is bipartite (resp. nonbipartite).*

Proof. Use Lemmas 13.36, 13.37. \square

Lemma 16.4. *Assume that A, B, C is bipartite (resp. nonbipartite). Let α, β, γ denote nonzero scalars in \mathbb{F} . Then the LR triple $\alpha A, \beta B, \gamma C$ is bipartite (resp. nonbipartite).*

Proof. Use Lemma 13.35. \square

Lemma 16.5. *Assume that A, B, C is nonbipartite. Then $d \geq 1$. Moreover each of*

$$\alpha_1, \quad \alpha'_1, \quad \alpha''_1, \quad \beta_1, \quad \beta'_1, \quad \beta''_1 \quad (113)$$

is nonzero.

Proof. We have $d \geq 1$ by Lemma 16.2. We show $\alpha_1 \neq 0$. Suppose $\alpha_1 = 0$. By Corollary 14.5 we obtain $\alpha'_1 = 0$ and $\alpha''_1 = 0$. Observe that $\beta_1 = -\alpha_1 = 0$. Similarly $\beta'_1 = 0$ and $\beta''_1 = 0$. Now by Proposition 14.1 each of a_i, a'_i, a''_i is zero for $0 \leq i \leq d$. Now A, B, C is bipartite, for a contradiction. We have shown that $\alpha_1 \neq 0$; by Lemma 16.3 the other scalars in (113) are nonzero. \square

Lemma 16.6. *Assume that A, B, C is bipartite. Then d is even. Moreover for $0 \leq i \leq d$, each of*

$$\alpha_i, \quad \alpha'_i, \quad \alpha''_i, \quad \beta_i, \quad \beta'_i, \quad \beta''_i \quad (114)$$

is zero if i is odd and nonzero if i is even.

Proof. We claim that α_i is zero if i is odd and nonzero if i is even. We prove the claim by induction on i . The claim holds for $i = 0$, since $\alpha_0 = 1$. The claim holds for $i = 1$, since $\alpha_1 = 0$ by Corollary 14.4 and our assumption that A, B, C is bipartite. The claim holds for $2 \leq i \leq d$ by Proposition 15.12(i) and induction. The claim is proven. By Lemma 16.3 the other scalars in (114) are zero if i is odd and nonzero if i is even. The diameter d must be even by the first assertion of Lemma 13.62. \square

As we continue to discuss LR triples, we will often treat the bipartite and nonbipartite cases separately. For the next few results, we consider the nonbipartite case.

Lemma 16.7. *Assume that A, B, C is nonbipartite. Then for $0 \leq i \leq d$,*

$$\frac{\alpha_i}{\alpha_1^i} = \frac{\alpha'_i}{(\alpha'_1)^i} = \frac{\alpha''_i}{(\alpha''_1)^i}, \quad \frac{\beta_i}{\beta_1^i} = \frac{\beta'_i}{(\beta'_1)^i} = \frac{\beta''_i}{(\beta''_1)^i}. \quad (115)$$

Proof. By Lemma 13.63 and Corollary 14.5. □

Lemma 16.8. *Assume that A, B, C is nonbipartite. Let α, β, γ denote nonzero scalars in \mathbb{F} . Then the following are equivalent:*

- (i) *the LR triples A, B, C and $\alpha A, \beta B, \gamma C$ are isomorphic;*
- (ii) $\alpha = \beta = \gamma = 1$.

Proof. Use Lemma 13.48. □

We turn our attention to bipartite LR triples.

Lemma 16.9. *Assume that A, B, C is bipartite and nontrivial. Then*

- (i) $\alpha_1, \alpha'_1, \alpha''_1$ and $\beta_1, \beta'_1, \beta''_1$ are all zero;
- (ii) $\alpha_2, \alpha'_2, \alpha''_2$ and $\beta_2, \beta'_2, \beta''_2$ are all nonzero;
- (iii) *We have*

$$\beta_2 = -\alpha_2, \quad \beta'_2 = -\alpha'_2, \quad \beta''_2 = -\alpha''_2. \quad (116)$$

Proof. (i), (ii) By Lemma 16.6.

(iii) From above Lemma 12.5. □

Lemma 16.10. *A bipartite LR triple is uniquely determined up to isomorphism by its parameter array.*

Proof. By Proposition 13.40 and Definition 16.1. □

Lemma 16.11. *Assume that A, B, C is bipartite. Let α, β, γ denote nonzero scalars in \mathbb{F} . Then the following are equivalent:*

- (i) *the LR triples A, B, C and $\alpha A, \beta B, \gamma C$ are isomorphic;*
- (ii) $\alpha = \beta = \gamma \in \{1, -1\}$.

Proof. Use Lemmas 13.8, 16.10. □

Lemma 16.12. *Assume that A, B, C is bipartite, so that $d = 2m$ is even.*

(i) The following subspaces are equal:

$$\sum_{j=0}^m E_{2j} V, \quad \sum_{j=0}^m E'_{2j} V, \quad \sum_{j=0}^m E''_{2j} V. \quad (117)$$

(ii) The following subspaces are equal:

$$\sum_{j=0}^{m-1} E_{2j+1} V, \quad \sum_{j=0}^{m-1} E'_{2j+1} V, \quad \sum_{j=0}^{m-1} E''_{2j+1} V. \quad (118)$$

(iii) Let V_{out} and V_{in} denote the common values of (117) and (118), respectively. Then

$$V = V_{\text{out}} + V_{\text{in}} \quad (\text{direct sum}). \quad (119)$$

(iv) We have

$$\dim(V_{\text{out}}) = m + 1, \quad \dim(V_{\text{in}}) = m. \quad (120)$$

Proof. (i) Denote the sequence in (117) by U, U', U'' . We show $U' = U$. The sequence $\{E_i V\}_{i=0}^d$ is the (A, B) -decomposition of V . Therefore $\{E_{d-i} V\}_{i=0}^d$ is the (B, A) -decomposition of V . The sequence $\{E'_i V\}_{i=0}^d$ is the (B, C) -decomposition of V . Let $\{u_i\}_{i=0}^d$ denote a (B, A) -basis for V . Let $\{v_i\}_{i=0}^d$ denote a compatible (B, C) -basis for V . Thus for $0 \leq i \leq d$, u_i (resp. v_i) is a basis for $E_{d-i} V$ (resp. $E'_i V$). Consequently $\{u_{2j}\}_{j=0}^m$ and $\{v_{2j}\}_{j=0}^m$ are bases for U and U' , respectively. The matrix T'' from Definition 13.44(iii) is the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$. By construction T'' is upper triangular with (i, r) -entry α''_{r-i} for $0 \leq i \leq r \leq d$. By Lemma 16.6 the scalars $\alpha''_1, \alpha''_3, \dots, \alpha''_{d-1}$ are zero. So the (i, r) -entry of T'' is zero if $r - i$ is odd ($0 \leq i \leq r \leq d$). By these comments $v_{2j} \in U$ for $0 \leq j \leq m$. Therefore $U' \subseteq U$. In this inclusion each side has dimension $m + 1$, so $U' = U$. One similarly shows that $U'' = U'$.

(ii) Similar to the proof of (i) above.

(iii), (iv) The sequence $\{E_i V\}_{i=0}^d$ is a decomposition of V . □

Definition 16.13. Referring to Lemma 16.12 and following Definition 8.6, we call V_{out} (resp. V_{in}) the *outer part* (resp. *inner part*) of V with respect to A, B, C .

Lemma 16.14. Assume that A, B, C is bipartite. Then $V_{\text{out}} \neq 0$ and $V_{\text{in}} \neq V$. Moreover the following are equivalent: (i) A, B is trivial; (ii) $V_{\text{out}} = V$; (iii) $V_{\text{in}} = 0$.

Proof. By Lemma 8.3. □

Lemma 16.15. Assume that A, B, C is bipartite. Then

$$\begin{aligned} AV_{\text{out}} &= V_{\text{in}}, & BV_{\text{out}} &= V_{\text{in}}, & CV_{\text{out}} &= V_{\text{in}}, \\ AV_{\text{in}} &\subseteq V_{\text{out}}, & BV_{\text{in}} &\subseteq V_{\text{out}}, & CV_{\text{in}} &\subseteq V_{\text{out}}. \end{aligned}$$

Moreover

$$\begin{aligned} A^2 V_{\text{out}} &\subseteq V_{\text{out}}, & B^2 V_{\text{out}} &\subseteq V_{\text{out}}, & C^2 V_{\text{out}} &\subseteq V_{\text{out}}, \\ A^2 V_{\text{in}} &\subseteq V_{\text{in}}, & B^2 V_{\text{in}} &\subseteq V_{\text{in}}, & C^2 V_{\text{in}} &\subseteq V_{\text{in}}. \end{aligned}$$

Proof. Use Lemma 8.5. □

Definition 16.16. For notational convenience define

$$t_i = \frac{\varphi'_{d-i+1}\varphi''_{d-i+1}}{\varphi_i}, \quad t'_i = \frac{\varphi''_{d-i+1}\varphi_{d-i+1}}{\varphi'_i}, \quad t''_i = \frac{\varphi_{d-i+1}\varphi'_{d-i+1}}{\varphi''_i}$$

for $1 \leq i \leq d$, and

$$t_0 = 0, \quad t'_0 = 0, \quad t''_0 = 0, \quad t_{d+1} = 0, \quad t'_{d+1} = 0, \quad t''_{d+1} = 0.$$

The following two lemmas are obtained by routine computation.

Lemma 16.17. *Assume that A, B, C is bipartite. Then the action of A^2, B^2, C^2 on V_{out} is an LR triple with diameter $m = d/2$. For this LR triple,*

(i) *the parameter array is*

$$(\{\varphi_{2j-1}\varphi_{2j}\}_{j=1}^m; \{\varphi'_{2j-1}\varphi'_{2j}\}_{j=1}^m; \{\varphi''_{2j-1}\varphi''_{2j}\}_{j=1}^m);$$

(ii) *the idempotent data is*

$$(\{E_{2j}\}_{j=0}^m; \{E'_{2j}\}_{j=0}^m; \{E''_{2j}\}_{j=0}^m);$$

(iii) *the trace data is (using the notation of Definition 16.16)*

$$(\{t_{2j} + t_{2j+1}\}_{j=0}^m; \{t'_{2j} + t'_{2j+1}\}_{j=0}^m; \{t''_{2j} + t''_{2j+1}\}_{j=0}^m);$$

(iv) *the Toeplitz data is*

$$(\{\alpha_{2j}\}_{j=0}^m, \{\beta_{2j}\}_{j=0}^m; \{\alpha'_{2j}\}_{j=0}^m, \{\beta'_{2j}\}_{j=0}^m; \{\alpha''_{2j}\}_{j=0}^m, \{\beta''_{2j}\}_{j=0}^m).$$

Lemma 16.18. *Assume that A, B, C is bipartite and nontrivial. Then the action of A^2, B^2, C^2 on V_{in} is an LR triple with diameter $m - 1$, where $m = d/2$. For this LR triple,*

(i) *the parameter array is*

$$(\{\varphi_{2j}\varphi_{2j+1}\}_{j=1}^{m-1}; \{\varphi'_{2j}\varphi'_{2j+1}\}_{j=1}^{m-1}; \{\varphi''_{2j}\varphi''_{2j+1}\}_{j=1}^{m-1});$$

(ii) *the idempotent data is*

$$(\{E_{2j+1}\}_{j=0}^{m-1}; \{E'_{2j+1}\}_{j=0}^{m-1}; \{E''_{2j+1}\}_{j=0}^{m-1});$$

(iii) *the trace data is (using the notation of Definition 16.16)*

$$(\{t_{2j+1} + t_{2j+2}\}_{j=0}^{m-1}; \{t'_{2j+1} + t'_{2j+2}\}_{j=0}^{m-1}; \{t''_{2j+1} + t''_{2j+2}\}_{j=0}^{m-1});$$

(iv) the Toeplitz data is

$$(\{\alpha_{2j}\}_{j=0}^{m-1}, \{\beta_{2j}\}_{j=0}^{m-1}; \{\alpha'_{2j}\}_{j=0}^{m-1}, \{\beta'_{2j}\}_{j=0}^{m-1}; \{\alpha''_{2j}\}_{j=0}^{m-1}, \{\beta''_{2j}\}_{j=0}^{m-1}).$$

Lemma 16.19. Assume that A, B, C is bipartite, and consider the action of A^2, B^2, C^2 on V_{out} .

- (i) Assume that A, B, C is trivial. Then so is the action of A^2, B^2, C^2 on V_{out} .
- (ii) Assume that A, B, C is nontrivial. Then the action of A^2, B^2, C^2 on V_{out} is nonbipartite.

Proof. (i) By Example 13.3 and Lemma 16.14.

(ii) Evaluate the Toeplitz data in Lemma 16.17(iv) using Lemmas 16.5, 16.6. \square

Lemma 16.20. Assume that A, B, C is bipartite and nontrivial. Consider the action of A^2, B^2, C^2 on V_{in} .

- (i) Assume that $d = 2$. Then the action of A^2, B^2, C^2 on V_{in} is trivial.
- (ii) Assume that $d \geq 4$. Then the action of A^2, B^2, C^2 on V_{in} is nonbipartite.

Proof. (i) By Example 13.3 and Lemma 16.14.

(ii) Evaluate the Toeplitz data in Lemma 16.18(iv) using Lemmas 16.5, 16.6. \square

Lemma 16.21. Assume that A, B, C is bipartite and nontrivial. Then for $2 \leq i \leq d$,

$$\frac{\alpha_2}{\varphi_{i-1}\varphi_i} = \frac{\alpha'_2}{\varphi'_{i-1}\varphi'_i} = \frac{\alpha''_2}{\varphi''_{i-1}\varphi''_i}, \quad \frac{\beta_2}{\varphi_{i-1}\varphi_i} = \frac{\beta'_2}{\varphi'_{i-1}\varphi'_i} = \frac{\beta''_2}{\varphi''_{i-1}\varphi''_i}. \quad (121)$$

Proof. Apply Corollary 14.5 to the LR triples in Lemmas 16.17, 16.18. \square

Lemma 16.22. Assume that A, B, C is bipartite and nontrivial. Then the following (i), (ii) hold for $1 \leq i, j \leq d$.

- (i) Assume that i, j have opposite parity. Then

$$\frac{\alpha_2}{\varphi_i\varphi_j} = \frac{\alpha'_2}{\varphi'_i\varphi'_j} = \frac{\alpha''_2}{\varphi''_i\varphi''_j}, \quad \frac{\beta_2}{\varphi_i\varphi_j} = \frac{\beta'_2}{\varphi'_i\varphi'_j} = \frac{\beta''_2}{\varphi''_i\varphi''_j}. \quad (122)$$

- (ii) Assume that i, j have the same parity. Then

$$\frac{\varphi_i}{\varphi_j} = \frac{\varphi'_i}{\varphi'_j} = \frac{\varphi''_i}{\varphi''_j}. \quad (123)$$

Proof. We have a preliminary remark. For $2 \leq k \leq d$ define $x_k = \alpha_2(\varphi_{k-1}\varphi_k)^{-1}$, and note that $x_k = x'_k = x''_k$ by Lemma 16.21.

- (i) We may assume without loss that $i < j$. Observe that

$$\frac{\alpha_2}{\varphi_i\varphi_j} = \frac{x_{i+1}x_{i+3}\cdots x_j}{x_{i+2}x_{i+4}\cdots x_{j-1}}.$$

By this and the preliminary remark, we obtain the equations on the left in (122). The equations on the right in (122) are similarly obtained.

(ii) We may assume without loss that $i < j$. Observe that

$$\frac{\varphi_i}{\varphi_j} = \frac{x_{i+2}x_{i+4} \cdots x_j}{x_{i+1}x_{i+3} \cdots x_{j-1}}.$$

By this and the preliminary remark, we obtain the equations (123). \square

Lemma 16.23. *Assume that A, B, C is bipartite and nontrivial. Then for $0 \leq j \leq d/2$,*

$$\frac{\alpha_{2j}}{\alpha_2^j} = \frac{\alpha'_{2j}}{(\alpha'_2)^j} = \frac{\alpha''_{2j}}{(\alpha''_2)^j}, \quad \frac{\beta_{2j}}{\beta_2^j} = \frac{\beta'_{2j}}{(\beta'_2)^j} = \frac{\beta''_{2j}}{(\beta''_2)^j}. \quad (124)$$

Proof. Apply Lemma 16.7 to the LR triple in Lemma 16.17. This LR triple is nonbipartite by Lemma 16.19(ii). \square

Definition 16.24. Assume that A, B, C is bipartite. An ordered pair of elements chosen from A, B, C form an LR pair; consider the corresponding projector map from Definition 9.1. By Lemma 16.12 this projector is independent of the choice; denote this common projector by J . We call J the *projector* for A, B, C .

In Section 9 we discussed in detail the projector map for LR pairs. We now adapt a few points to LR triples.

Lemma 16.25. *Assume that A, B, C is bipartite. Then its projector map J is nonzero. If A, B, C is trivial then $J = I$. If A, B, C is nontrivial then J, I are linearly independent over \mathbb{F} .*

Proof. By Lemma 9.2 and Definition 16.24. \square

Lemma 16.26. *Assume that A, B, C is bipartite. Then its projector map J satisfies*

$$J = \sum_{j=0}^{d/2} E_{2j} = \sum_{j=0}^{d/2} E'_{2j} = \sum_{j=0}^{d/2} E''_{2j}.$$

Moreover $J^2 = J$. Also, J commutes with each of E_i, E'_i, E''_i for $0 \leq i \leq d$.

Proof. By Lemma 9.3 and Definition 16.24. \square

Lemma 16.27. *Assume that A, B, C is bipartite. Then its projector map J satisfies*

$$A = AJ + JA, \quad B = BJ + JB, \quad C = CJ + JC.$$

Proof. By Lemma 9.9(iii) and Definition 16.24. \square

Lemma 16.28. *Assume that A, B, C is bipartite. With respect to any of the 12 bases (51)–(53), the matrix representing J is $\text{diag}(1, 0, 1, 0, \dots, 0, 1)$.*

Proof. By construction and linear algebra. \square

The following definition is motivated by Definition 8.9.

Definition 16.29. Assume that A, B, C is bipartite. Define

$$A_{\text{out}}, \quad A_{\text{in}}, \quad B_{\text{out}}, \quad B_{\text{in}}, \quad C_{\text{out}}, \quad C_{\text{in}} \quad (125)$$

in $\text{End}(V)$ as follows. The map A_{out} acts on V_{out} as A , and on V_{in} as zero. The map A_{in} acts on V_{in} as A , and on V_{out} as zero. The other maps in (125) are similarly defined. By construction

$$A = A_{\text{out}} + A_{\text{in}}, \quad B = B_{\text{out}} + B_{\text{in}}, \quad C = C_{\text{out}} + C_{\text{in}}.$$

Lemma 16.30. Assume that A, B, C is bipartite. Then

$$\begin{aligned} A_{\text{out}} &= AJ = (I - J)A, & A_{\text{in}} &= JA = A(I - J), \\ B_{\text{out}} &= BJ = (I - J)B, & B_{\text{in}} &= JB = B(I - J), \\ C_{\text{out}} &= CJ = (I - J)C, & C_{\text{in}} &= JC = C(I - J). \end{aligned}$$

Proof. By Lemma 9.9(i),(ii) and Definition 16.24. □

Lemma 16.31. Assume that A, B, C is bipartite. Let

$$\alpha_{\text{out}}, \quad \alpha_{\text{in}}, \quad \beta_{\text{out}}, \quad \beta_{\text{in}}, \quad \gamma_{\text{out}}, \quad \gamma_{\text{in}} \quad (126)$$

denote nonzero scalars in \mathbb{F} . Then the sequence

$$\alpha_{\text{out}}A_{\text{out}} + \alpha_{\text{in}}A_{\text{in}}, \quad \beta_{\text{out}}B_{\text{out}} + \beta_{\text{in}}B_{\text{in}}, \quad \gamma_{\text{out}}C_{\text{out}} + \gamma_{\text{in}}C_{\text{in}} \quad (127)$$

is a bipartite LR triple on V .

Proof. By construction. □

Our next goal is to obtain the parameter array, idempotent data, and Toeplitz data for the LR triple in (127). The following definition is for notational convenience.

Definition 16.32. Adopt the assumptions and notation of Lemma 16.31. For $1 \leq i \leq d$ define

$$\begin{aligned} f_i &= \alpha_{\text{out}}\beta_{\text{in}}, & f'_i &= \beta_{\text{out}}\gamma_{\text{in}}, & f''_i &= \gamma_{\text{out}}\alpha_{\text{in}} & (\text{if } i \text{ is even}), \\ f_i &= \alpha_{\text{in}}\beta_{\text{out}}, & f'_i &= \beta_{\text{in}}\gamma_{\text{out}}, & f''_i &= \gamma_{\text{in}}\alpha_{\text{out}} & (\text{if } i \text{ is odd}). \end{aligned}$$

Also define

$$\begin{aligned} g_i &= (\alpha_{\text{out}}\alpha_{\text{in}})^{-i/2}, & g'_i &= (\beta_{\text{out}}\beta_{\text{in}})^{-i/2}, & g''_i &= (\gamma_{\text{out}}\gamma_{\text{in}})^{-i/2} & (\text{if } i \text{ is even}), \\ g_i &= 0, & g'_i &= 0, & g''_i &= 0 & (\text{if } i \text{ is odd}). \end{aligned}$$

Lemma 16.33. Referring to Lemma 16.31, let the nonzero scalars (126) be given, and consider the LR triple in (127). For this LR triple,

(i) the parameter array is (using the notation of Definition 16.32)

$$(\{\varphi_i f_i\}_{i=1}^d; \{\varphi'_i f'_i\}_{i=1}^d; \{\varphi''_i f''_i\}_{i=1}^d);$$

(ii) the idempotent data is equal to the idempotent data for A, B, C ;

(iii) the Toeplitz data is (using the notation of Definition 16.32)

$$(\{\alpha_i g''_i\}_{i=0}^d, \{\beta_i g''_i\}_{i=0}^d; \{\alpha'_i g_i\}_{i=0}^d, \{\beta'_i g_i\}_{i=0}^d; \{\alpha''_i g'_i\}_{i=0}^d, \{\beta''_i g'_i\}_{i=0}^d).$$

Proof. (i) Use Lemma 10.7.

(ii) Similar to the proof of Lemma 13.22.

(iii) Similar to the proof of Lemma 13.48. \square

Definition 16.34. Assume that A, B, C is bipartite. Let A', B', C' denote a bipartite LR triple on V . Then A, B, C and A', B', C' will be called *biassociate* whenever there exist nonzero scalars α, β, γ in \mathbb{F} such that

$$A' = \alpha A_{\text{out}} + A_{\text{in}}, \quad B' = \beta B_{\text{out}} + B_{\text{in}}, \quad C' = \gamma C_{\text{out}} + C_{\text{in}}.$$

Biassociativity is an equivalence relation.

Lemma 16.35. Assume that A, B, C is bipartite. Let A', B', C' denote a bipartite LR triple over \mathbb{F} . Then the following are equivalent:

- (i) there exists a bipartite LR triple over \mathbb{F} that is biassociate to A, B, C and isomorphic to A', B', C' ;
- (ii) there exists a bipartite LR triple over \mathbb{F} that is isomorphic to A, B, C and biassociate to A', B', C' .

Proof. Similar to the proof of Lemma 10.3. \square

Definition 16.36. Assume that A, B, C is bipartite. Let A', B', C' denote a bipartite LR triple over \mathbb{F} . Then A, B, C and A', B', C' will be called *bisimilar* whenever the equivalent conditions (i), (ii) hold in Lemma 16.35. Bisimilarity is an equivalence relation.

17 Equitable LR triples

Throughout this section the following notation is in effect. Let V denote a vector space over \mathbb{F} with dimension $d + 1$. Let A, B, C denote an LR triple on V , with parameter array (44), idempotent data (48), trace data (50), and Toeplitz data (54). We describe a condition on A, B, C called equitable.

Definition 17.1. The LR triple A, B, C is called *equitable* whenever $\alpha_i = \alpha'_i = \alpha''_i$ for $0 \leq i \leq d$.

Lemma 17.2. If A, B, C is trivial, then it is equitable.

Proof. Recall that $\alpha_0 = \alpha'_0 = \alpha''_0 = 1$. □

Lemma 17.3. *Assume that A, B, C is equitable. Then $\beta_i = \beta'_i = \beta''_i$ for $0 \leq i \leq d$.*

Proof. Refer to Definitions 13.44, 13.45. The matrices T, T', T'' coincide, so their inverses coincide. The result follows. □

Lemma 17.4. *If A, B, C is equitable, then so are its relatives.*

Proof. By Lemmas 13.49, 13.51, 17.3 and Definition 17.1. □

As we investigate the equitable property, we will treat the bipartite and nonbipartite cases separately. We begin with the nonbipartite case.

Lemma 17.5. *Assume that A, B, C is nonbipartite. Then A, B, C is equitable if and only if $\alpha_1 = \alpha'_1 = \alpha''_1$.*

Proof. By Lemma 16.7 and Definition 17.1. □

Lemma 17.6. *Assume that A, B, C is nonbipartite and equitable. Then the following hold:*

- (i) $\varphi_i = \varphi'_i = \varphi''_i$ for $1 \leq i \leq d$;
- (ii) $a_i = a'_i = a''_i = \alpha_1(\varphi_{d-i+1} - \varphi_{d-i})$ for $0 \leq i \leq d$.

Proof. (i) By Corollary 14.5 and Lemma 16.5.

(ii) Use (57) and Proposition 14.1. □

The following definition is for later use.

Definition 17.7. Assume that A, B, C is nonbipartite and equitable. Define

$$\rho_i = \frac{\varphi_{i+1}}{\varphi_{d-i}} \quad (0 \leq i \leq d-1). \quad (128)$$

Note by Lemma 17.6(i) that $\rho_i = \rho'_i = \rho''_i$ for $0 \leq i \leq d-1$.

Lemma 17.8. *Assume that A, B, C is nonbipartite and equitable. Then $\rho_i \rho_{d-i-1} = 1$ for $0 \leq i \leq d-1$.*

Proof. By Definition 17.7. □

Lemma 17.9. *Assume that A, B, C is nonbipartite. Let α, β, γ denote nonzero scalars in \mathbb{F} . Then the following are equivalent:*

- (i) *the LR triple $\alpha A, \beta B, \gamma C$ is equitable;*
- (ii) $\alpha/\alpha'_1 = \beta/\alpha''_1 = \gamma/\alpha_1$.

Proof. By Lemmas 13.48, 17.5. □

Lemma 17.10. *Assume that A, B, C is nonbipartite. Then there exists an equitable LR triple on V that is associate to A, B, C .*

Proof. By Definition 13.10 and Lemma 17.9. \square

We turn our attention to bipartite LR triples.

Lemma 17.11. *Assume that A, B, C is bipartite and nontrivial. Then A, B, C is equitable if and only if $\alpha_2 = \alpha'_2 = \alpha''_2$.*

Proof. By Lemmas 16.6, 16.23 and Definition 17.1. \square

Lemma 17.12. *Assume that A, B, C is bipartite, nontrivial, and equitable. Then $\varphi_{i-1}\varphi_i = \varphi'_{i-1}\varphi'_i = \varphi''_{i-1}\varphi''_i$ for $2 \leq i \leq d$.*

Proof. By Lemma 16.9(ii) and Lemma 16.21. \square

Lemma 17.13. *Assume that A, B, C is bipartite and equitable. Then the following (i), (ii) hold for $0 \leq i \leq d$.*

(i) *For i even,*

$$\begin{aligned}\varphi_1\varphi_2 \cdots \varphi_i &= \varphi'_1\varphi'_2 \cdots \varphi'_i = \varphi''_1\varphi''_2 \cdots \varphi''_i, \\ \varphi_d\varphi_{d-1} \cdots \varphi_{d-i+1} &= \varphi'_d\varphi'_{d-1} \cdots \varphi'_{d-i+1} = \varphi''_d\varphi''_{d-1} \cdots \varphi''_{d-i+1}.\end{aligned}$$

(ii) *For i odd,*

$$\begin{aligned}\varphi_2\varphi_3 \cdots \varphi_i &= \varphi'_2\varphi'_3 \cdots \varphi'_i = \varphi''_2\varphi''_3 \cdots \varphi''_i, \\ \varphi_{d-1}\varphi_{d-2} \cdots \varphi_{d-i+1} &= \varphi'_{d-1}\varphi'_{d-2} \cdots \varphi'_{d-i+1} = \varphi''_{d-1}\varphi''_{d-2} \cdots \varphi''_{d-i+1}.\end{aligned}$$

Proof. By Lemma 17.12 and since d is even. \square

Lemma 17.14. *Assume that A, B, C is bipartite and equitable. Then for $0 \leq i \leq d-1$,*

$$\frac{\varphi'_{i+1}}{\varphi''_{d-i}} = \frac{\varphi''_{i+1}}{\varphi'_{d-i}}, \quad \frac{\varphi''_{i+1}}{\varphi_{d-i}} = \frac{\varphi_{i+1}}{\varphi''_{d-i}}, \quad \frac{\varphi_{i+1}}{\varphi'_{d-i}} = \frac{\varphi'_{i+1}}{\varphi_{d-i}}.$$

Proof. Assume that A, B, C is nontrivial; otherwise there is nothing to prove. Now use Lemma 16.22(i) with $j = d - i + 1$. The integers i, j have opposite parity since d is even. \square

Definition 17.15. *Assume that A, B, C is bipartite and equitable. Then for $0 \leq i \leq d-1$ define*

$$\begin{aligned}\rho_i &= \frac{\varphi'_{i+1}}{\varphi''_{d-i}} = \frac{\varphi''_{i+1}}{\varphi'_{d-i}}, \\ \rho'_i &= \frac{\varphi''_{i+1}}{\varphi_{d-i}} = \frac{\varphi_{i+1}}{\varphi''_{d-i}}, \\ \rho''_i &= \frac{\varphi_{i+1}}{\varphi'_{d-i}} = \frac{\varphi'_{i+1}}{\varphi_{d-i}}.\end{aligned}$$

We emphasize that for $d \geq 1$,

$$\rho_0 = \frac{\varphi'_1}{\varphi''_d} = \frac{\varphi''_1}{\varphi'_d}, \quad \rho'_0 = \frac{\varphi''_1}{\varphi_d} = \frac{\varphi_1}{\varphi''_d}, \quad \rho''_0 = \frac{\varphi_1}{\varphi'_d} = \frac{\varphi'_1}{\varphi_d}. \quad (129)$$

Lemma 17.16. Assume that A, B, C is bipartite and equitable. Then for $0 \leq i \leq d-1$,

$$\rho_i \rho_{d-i-1} = 1, \quad \rho'_i \rho'_{d-i-1} = 1, \quad \rho''_i \rho''_{d-i-1} = 1. \quad (130)$$

Proof. By Definition 17.15. \square

Lemma 17.17. Assume that A, B, C is bipartite, nontrivial, and equitable. Then the following (i)–(iii) hold:

(i) for $1 \leq i \leq d$,

$$\begin{aligned} \frac{\varphi_i}{\rho_0} &= \frac{\varphi'_i}{\rho'_0} = \frac{\varphi''_i}{\rho''_0} && \text{if } i \text{ is even,} \\ \varphi_i \rho_0 &= \varphi'_i \rho'_0 = \varphi''_i \rho''_0 && \text{if } i \text{ is odd;} \end{aligned}$$

(ii) for $0 \leq i \leq d$,

$$\begin{aligned} \rho_0 \rho_1 \cdots \rho_{i-1} &= \rho'_0 \rho'_1 \cdots \rho'_{i-1} = \rho''_0 \rho''_1 \cdots \rho''_{i-1} && \text{if } i \text{ is even,} \\ \rho_1 \rho_2 \cdots \rho_{i-1} &= \rho'_1 \rho'_2 \cdots \rho'_{i-1} = \rho''_1 \rho''_2 \cdots \rho''_{i-1} && \text{if } i \text{ is odd;} \end{aligned}$$

(iii) for $0 \leq i \leq d-1$,

$$\begin{aligned} \frac{\rho_i}{\rho_0} &= \frac{\rho'_i}{\rho'_0} = \frac{\rho''_i}{\rho''_0} && \text{if } i \text{ is even,} \\ \rho_i \rho_0 &= \rho'_i \rho'_0 = \rho''_i \rho''_0 && \text{if } i \text{ is odd.} \end{aligned}$$

Proof. Use Lemma 16.22 and Definition 17.15. \square

Lemma 17.18. Assume that A, B, C is bipartite and equitable. Then:

(i) the action of A^2, B^2, C^2 on V_{out} is equitable;

(ii) for A, B, C nontrivial the action of A^2, B^2, C^2 on V_{in} is equitable.

Proof. (i) By Lemma 16.17(iv) and Definition 17.1.

(ii) By Lemma 16.18(iv) and Definition 17.1. \square

Lemma 17.19. Referring to Lemma 16.31, assume that A, B, C is nontrivial, and let the nonzero scalars (126) be given. Then the LR triple (127) is equitable if and only if

$$\alpha_{\text{out}} \alpha_{\text{in}} / \alpha'_2 = \beta_{\text{out}} \beta_{\text{in}} / \alpha''_2 = \gamma_{\text{out}} \gamma_{\text{in}} / \alpha_2. \quad (131)$$

Proof. By Lemma 16.33(iii) and Definition 17.1. \square

Lemma 17.20. Assume that A, B, C is bipartite and nontrivial. Then there exists an equitable LR triple on V that is biassociate to A, B, C .

Proof. By Definition 16.34 and Lemma 17.19. \square

We have a comment about general LR triples, bipartite or not.

Lemma 17.21. Assume that A, B, C is equitable. Then

$$\varphi_1 \varphi_2 \cdots \varphi_d = \varphi'_1 \varphi'_2 \cdots \varphi'_d = \varphi''_1 \varphi''_2 \cdots \varphi''_d. \quad (132)$$

Proof. For A, B, C nonbipartite, the result follows from Lemma 17.6(i). For A, B, C bipartite, the result follows from Lemma 17.13(i) and since d is even. \square

18 Normalized LR triples

Throughout this section the following notation is in effect. Let V denote a vector space over \mathbb{F} with dimension $d + 1$. Let A, B, C denote an LR triple on V , with parameter array (44), idempotent data (48), trace data (50), and Toeplitz data (54). We describe a condition on A, B, C called normalized. The condition is defined a bit differently in the trivial, nonbipartite, and bipartite nontrivial cases. We first dispense with the trivial case.

Definition 18.1. Assume that A, B, C is trivial. Then we declare A, B, C to be *normalized*.

Definition 18.2. Assume that A, B, C is nonbipartite. Then A, B, C is called *normalized* whenever

$$\alpha_1 = 1, \quad \alpha'_1 = 1, \quad \alpha''_1 = 1.$$

Lemma 18.3. Assume that A, B, C is nonbipartite and normalized. Then A, B, C is equitable.

Proof. By Lemma 17.5 and since $\alpha_1 = \alpha'_1 = \alpha''_1$. □

Lemma 18.4. Assume that A, B, C is nonbipartite and normalized. Then so are its p -relatives.

Proof. By Definition 13.14 and Lemmas 13.49, 13.51. □

A nonbipartite LR triple can be normalized as follows.

Lemma 18.5. Assume that A, B, C is nonbipartite. Let α, β, γ denote nonzero scalars in \mathbb{F} . Then the LR triple $\alpha A, \beta B, \gamma C$ is normalized if and only if

$$\alpha = \alpha'_1, \quad \beta = \alpha''_1, \quad \gamma = \alpha_1.$$

Proof. Use Lemmas 13.48, 16.4 and Definition 18.2. □

Corollary 18.6. Assume that A, B, C is nonbipartite. Then there exists a unique sequence α, β, γ of nonzero scalars in \mathbb{F} such that $\alpha A, \beta B, \gamma C$ is normalized.

Proof. By Lemma 18.5. □

Corollary 18.7. Assume that A, B, C is nonbipartite. Then A, B, C is associate to a unique normalized nonbipartite LR triple over \mathbb{F} .

Proof. By Definition 13.10, Lemma 16.4, and Corollary 18.6. □

Lemma 18.8. Assume that A, B, C is nonbipartite and normalized. Then

$$\beta_1 = -1, \quad \beta'_1 = -1, \quad \beta''_1 = -1.$$

Proof. By (57) and Definition 18.2. □

Lemma 18.9. *Assume that A, B, C is nonbipartite and normalized. Then so is the LR triple $-C, -B, -A$.*

Proof. By Lemma 13.49 (row 4 of the table) along with Lemmas 18.5, 18.8. \square

Lemma 18.10. *Assume that A, B, C is nonbipartite and normalized. Then A, B, C is uniquely determined up to isomorphism by its parameter array.*

Proof. By Proposition 13.40 the LR triple A, B, C is uniquely determined up to isomorphism by its parameter array and trace data. The trace data is determined by the parameter array using Lemma 17.6(ii) and $\alpha_1 = 1$. The result follows. \square

We turn our attention to bipartite nontrivial LR triples.

Definition 18.11. Assume that A, B, C is bipartite and nontrivial. Then A, B, C is called *normalized* whenever

$$\alpha_2 = 1, \quad \alpha'_2 = 1, \quad \alpha''_2 = 1.$$

Lemma 18.12. *Assume that A, B, C is bipartite, nontrivial, and normalized. Then A, B, C is equitable.*

Proof. By Lemma 17.11 and since $\alpha_2 = \alpha'_2 = \alpha''_2$. \square

Lemma 18.13. *Assume that A, B, C is bipartite, nontrivial, and normalized. Then so are its p -relatives.*

Proof. By Definition 13.14 and Lemmas 13.49, 13.51. \square

A bipartite nontrivial LR triple can be normalized as follows.

Lemma 18.14. *Referring to Lemma 16.31, assume that A, B, C is nontrivial, and let the nonzero scalars (126) be given. Then the LR triple (127) is normalized if and only if*

$$\alpha_{\text{out}}\alpha_{\text{in}} = \alpha'_2, \quad \beta_{\text{out}}\beta_{\text{in}} = \alpha''_2, \quad \gamma_{\text{out}}\gamma_{\text{in}} = \alpha_2.$$

Proof. Use Lemma 16.33(iii) and Definition 18.11. \square

Corollary 18.15. *Assume that A, B, C is bipartite and nontrivial. Then there exists a unique sequence α, β, γ of nonzero scalars in \mathbb{F} such that*

$$\alpha A_{\text{out}} + A_{\text{in}}, \quad \beta B_{\text{out}} + B_{\text{in}}, \quad \gamma C_{\text{out}} + C_{\text{in}}$$

is normalized.

Proof. In Lemma 18.14 set $\alpha_{\text{in}} = 1, \beta_{\text{in}} = 1, \gamma_{\text{in}} = 1$ to see that $\alpha = \alpha'_2, \beta = \alpha''_2, \gamma = \alpha_2$ is the unique solution. \square

Corollary 18.16. *Assume that A, B, C is bipartite and nontrivial. Then A, B, C is biassociate to a unique bipartite normalized LR triple over \mathbb{F} .*

Proof. By Lemma 16.31, Definition 16.34, and Corollary 18.15. \square

Lemma 18.17. *Assume that A, B, C is bipartite, nontrivial, and normalized. Then*

$$\beta_2 = -1, \quad \beta'_2 = -1, \quad \beta''_2 = -1.$$

Proof. By (116) and Definition 18.11. \square

Lemma 18.18. *Assume that A, B, C is bipartite, nontrivial, and normalized. Then so is the LR triple*

$$C_{\text{out}} - C_{\text{in}}, \quad B_{\text{out}} - B_{\text{in}}, \quad A_{\text{out}} - A_{\text{in}}.$$

Proof. By Lemma 13.49 (row 4 of the table) and Lemmas 18.14, 18.17. \square

Lemma 18.19. *Assume that A, B, C is bipartite, nontrivial, and normalized. Then:*

- (i) *the action of A^2, B^2, C^2 on V_{out} is normalized;*
- (ii) *the action of A^2, B^2, C^2 on V_{in} is normalized.*

Proof. (i) Evaluate the Toeplitz data in Lemma 16.17(iv) using Lemma 16.19(ii) and Definition 18.2.

(ii) For $d = 2$, the action of A^2, B^2, C^2 on V_{in} is trivial and hence normalized. For $d \geq 4$, evaluate the Toeplitz data in Lemma 16.18(iv) using Lemma 16.20(ii) and Definition 18.2. \square

19 The idempotent centralizers for an LR triple

Throughout this section the following notation is in effect. Let V denote a vector space over \mathbb{F} with dimension $d + 1$. Let A, B, C denote an LR triple on V , with parameter array (44), idempotent data (48), trace data (50), and Toeplitz data (54). We discuss a type of element in $\text{End}(V)$ called an idempotent centralizer.

Definition 19.1. By an *idempotent centralizer* for A, B, C we mean an element in $\text{End}(V)$ that commutes with each of E_i, E'_i, E''_i for $0 \leq i \leq d$.

Lemma 19.2. *For $X \in \text{End}(V)$ the following are equivalent:*

- (i) *X is an idempotent centralizer for A, B, C ;*
- (ii) *for $0 \leq i \leq d$,*

$$XE_iV \subseteq E_iV, \quad XE'_iV \subseteq E'_iV, \quad XE''_iV \subseteq E''_iV.$$

Proof. By Definition 19.1 and linear algebra. \square

Example 19.3. The identity $I \in \text{End}(V)$ is an idempotent centralizer for A, B, C .

Definition 19.4. Let \mathcal{I} denote the set of idempotent centralizers for A, B, C . Note that \mathcal{I} is a subalgebra of the \mathbb{F} -algebra $\text{End}(V)$. We call \mathcal{I} the *idempotent centralizer algebra* for A, B, C .

Referring to Definition 19.4, our next goal is to display a basis for the \mathbb{F} -vector space \mathcal{I} . Recall the projector J from Definition 16.24.

Proposition 19.5. *The following (i)–(iii) hold.*

- (i) *Assume that A, B, C is trivial. Then I is a basis for \mathcal{I} .*
- (ii) *Assume that A, B, C is nonbipartite. Then I is a basis for \mathcal{I} .*
- (iii) *Assume that A, B, C is bipartite and nontrivial. Then I, J is a basis for \mathcal{I} .*

Proof. (i) Routine.

(ii), (iii) Assume that A, B, C is nontrivial. Let the set S consist of I (if A, B, C is nonbipartite) and I, J (if A, B, C is bipartite). We show that S is a basis for \mathcal{I} . By Lemma 16.25 and Lemma 16.26, S is a linearly independent subset of \mathcal{I} . We show that S spans \mathcal{I} . Let $\{u_i\}_{i=0}^d$ denote an (A, C) -basis of V , and let $\{v_i\}_{i=0}^d$ denote a compatible (A, B) -basis of V . The transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$ is the matrix T' from Definition 13.44. The matrix T' is upper triangular and Toeplitz, with parameters $\{\alpha'_i\}_{i=0}^d$. Let $X \in \mathcal{I}$. By Lemma 19.2 there exist scalars $\{r_i\}_{i=0}^d$ in \mathbb{F} such that $Xu_i = r_i u_i$ for $0 \leq i \leq d$. Also by Lemma 19.2, there exist scalars $\{s_i\}_{i=0}^d$ in \mathbb{F} such that $Xv_i = s_i v_i$ for $0 \leq i \leq d$. Let the matrix $M \in \text{Mat}_{d+1}(\mathbb{F})$ represent X with respect to $\{u_i\}_{i=0}^d$. Then M is diagonal with (i, i) -entry r_i for $0 \leq i \leq d$. Let the matrix $N \in \text{Mat}_{d+1}(\mathbb{F})$ represent X with respect to $\{v_i\}_{i=0}^d$. Then N is diagonal with (i, i) -entry s_i for $0 \leq i \leq d$. By linear algebra $MT' = T'N$. In this equation, for $0 \leq i \leq d$ compare the (i, i) -entry of each side, to obtain $r_i = s_i$. Until further notice assume that A, B, C is nonbipartite. Then $\alpha'_1 \neq 0$. In the equation $MT' = T'N$, for $1 \leq i \leq d$ compare the $(i-1, i)$ -entry of each side, to obtain $r_{i-1} = r_i$. So $r_i = r_0$ for $0 \leq i \leq d$. Consequently $X - r_0 I$ vanishes on u_i for $0 \leq i \leq d$. Therefore $X = r_0 I$. Next assume that A, B, C is bipartite. Then $\alpha'_1 = 0$ and $\alpha'_2 \neq 0$. In the equation $MT' = T'N$, for $2 \leq i \leq d$ compare the $(i-2, i)$ -entry of each side, to obtain $r_{i-2} = r_i$. For $0 \leq i \leq d$ we have $r_i = r_0$ (if i is even) and $r_i = r_1$ (if i is odd). Consequently $X - r_0 J - r_1(I - J)$ vanishes on u_i for $0 \leq i \leq d$. Therefore $X = r_0 J + r_1(I - J)$. We have shown that the set S spans \mathcal{I} . The result follows. \square

We have some comments about Proposition 19.5(iii).

Lemma 19.6. *Assume that A, B, C is bipartite and nontrivial. Let X denote an idempotent centralizer for A, B, C . Then*

- (i) $XV_{\text{out}} \subseteq V_{\text{out}}$ and $XV_{\text{in}} \subseteq V_{\text{in}}$;
- (ii) $XJ = JX$.

Proof. (i) By Lemmas 16.12, 19.2.

(ii) The map J acts on V_{out} as the identity, and on V_{in} as zero. The result follows from this and (i) above. \square

Definition 19.7. Assume that A, B, C is bipartite and nontrivial. Let X denote an idempotent centralizer for A, B, C . Then X is called *outer* (resp. *inner*) whenever X is zero on V_{in} (resp. V_{out}). Let \mathcal{I}_{out} (resp. \mathcal{I}_{in}) denote the set of outer (resp. inner) idempotent centralizers for A, B, C . Note that \mathcal{I}_{out} and \mathcal{I}_{in} are ideals in the algebra \mathcal{I} .

Proposition 19.8. Assume that A, B, C is bipartite and nontrivial. Then the following (i)–(iii) hold:

- (i) the sum $\mathcal{I} = \mathcal{I}_{\text{out}} + \mathcal{I}_{\text{in}}$ is direct;
- (ii) J is a basis for \mathcal{I}_{out} ;
- (iii) $I - J$ is a basis for \mathcal{I}_{in} .

Proof. By Definitions 9.1, 19.7 we find that $J \in \mathcal{I}_{\text{out}}$ and $I - J \in \mathcal{I}_{\text{in}}$. Also $\mathcal{I}_{\text{out}} \cap \mathcal{I}_{\text{in}} = 0$ by Definition 19.7 and since the sum $V = V_{\text{out}} + V_{\text{in}}$ is direct. The result follows in view of Proposition 19.5(iii). \square

20 The double lowering spaces for an LR triple

Throughout this section the following notation is in effect. Let V denote a vector space over \mathbb{F} with dimension $d + 1$. Let A, B, C denote an LR triple on V , with parameter array (44), idempotent data (48), trace data (50), and Toeplitz data (54). We discuss some subspaces of $\text{End}(V)$ called the double lowering spaces.

Definition 20.1. Let $\{V_i\}_{i=0}^d$ denote a decomposition of V . For $X \in \text{End}(V)$, we say that X *weakly lowers* $\{V_i\}_{i=0}^d$ whenever $XV_i \subseteq V_{i-1}$ for $1 \leq i \leq d$ and $XV_0 = 0$.

Definition 20.2. Let \overline{A} denote the set of elements in $\text{End}(V)$ that weakly lower both the (A, B) -decomposition of V and the (A, C) -decomposition of V . The sets $\overline{B}, \overline{C}$ are similarly defined. Note that $\overline{A}, \overline{B}, \overline{C}$ are subspaces of the \mathbb{F} -vector space $\text{End}(V)$. We call $\overline{A}, \overline{B}, \overline{C}$ the *double lowering spaces* for the LR triple A, B, C .

We now describe the \mathbb{F} -vector space \overline{A} ; similar results hold for \overline{B} and \overline{C} .

Theorem 20.3. The following (i)–(iii) hold.

- (i) Assume that A, B, C is trivial. Then $\overline{A} = 0$.
- (ii) Assume that A, B, C is nonbipartite. Then A is a basis for \overline{A} . Moreover \overline{A} has dimension 1.
- (iii) Assume that A, B, C is bipartite and nontrivial. Then $A_{\text{out}}, A_{\text{in}}$ form a basis for \overline{A} . Moreover \overline{A} has dimension 2.

Proof. (i) \overline{A} is zero on E_0V , and $E_0V = V$.

(ii), (iii) Assume that A, B, C is nontrivial. Let the set S consist of A (if A, B, C is nonbipartite) and $A_{\text{out}}, A_{\text{in}}$ (if A, B, C is bipartite). We show that S is a basis for \overline{A} . By Lemma 8.10 and the construction, S is a linearly independent subset of \overline{A} . We show that S spans

\overline{A} . Let $\{u_i\}_{i=0}^d$ denote an (A, C) -basis of V , and let $\{v_i\}_{i=0}^d$ denote a compatible (A, B) -basis of V . The transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$ is the matrix T' from Definition 13.44. The matrix T' is upper triangular and Toeplitz, with parameters $\{\alpha'_i\}_{i=0}^d$. Let $X \in \overline{A}$. The map X weakly lowers the (A, C) -decomposition of V , so there exist scalars $\{r_i\}_{i=1}^d$ in \mathbb{F} such that $Xu_i = r_i u_{i-1}$ for $1 \leq i \leq d$ and $Xu_0 = 0$. The map X weakly lowers the (A, B) -decomposition of V , so there exist scalars $\{s_i\}_{i=1}^d$ in \mathbb{F} such that $Xv_i = s_i v_{i-1}$ for $1 \leq i \leq d$ and $Xv_0 = 0$. Let the matrix $M \in \text{Mat}_{d+1}(\mathbb{F})$ represent X with respect to $\{u_i\}_{i=0}^d$. Then M has $(i-1, i)$ -entry r_i for $1 \leq i \leq d$, and all other entries 0. Let the matrix $N \in \text{Mat}_{d+1}(\mathbb{F})$ represent X with respect to $\{v_i\}_{i=0}^d$. Then N has $(i-1, i)$ -entry s_i for $1 \leq i \leq d$, and all other entries 0. By linear algebra $MT' = T'N$. In this equation, for $1 \leq i \leq d$ compare the $(i-1, i)$ -entry of each side, to obtain $r_i = s_i$. Until further notice assume that A, B, C is nonbipartite. Then $\alpha'_1 \neq 0$. In the equation $MT' = T'N$, for $1 \leq i \leq d-1$ compare the $(i-1, i+1)$ -entry of each side, to obtain $r_i = r_{i+1}$. So $r_i = r_1$ for $1 \leq i \leq d$. By construction $Au_i = u_{i-1}$ for $1 \leq i \leq d$ and $Au_0 = 0$. By these comments $X - r_1 A$ vanishes on u_i for $0 \leq i \leq d$. Therefore $X = r_1 A$. Next assume that A, B, C is bipartite. Then $\alpha'_1 = 0$ and $\alpha'_2 \neq 0$. In the equation $MT' = T'N$, for $2 \leq i \leq d-1$ compare the $(i-2, i+1)$ -entry of each side, to obtain $r_{i-1} = r_{i+1}$. For $1 \leq i \leq d$ we have $r_i = r_2$ (if i is even) and $r_i = r_1$ (if i is odd). For $0 \leq i \leq d$ define ε_i to be 0 (if i is even) and 1 (if i is odd). By construction $A_{\text{out}}u_i = (1 - \varepsilon_i)u_{i-1}$ for $1 \leq i \leq d$ and $A_{\text{out}}u_0 = 0$. Also by construction $A_{\text{in}}u_i = \varepsilon_i u_{i-1}$ for $1 \leq i \leq d$ and $A_{\text{in}}u_0 = 0$. By the above comments $X - r_1 A_{\text{in}} - r_2 A_{\text{out}}$ vanishes on u_i for $0 \leq i \leq d$. Therefore $X = r_1 A_{\text{in}} + r_2 A_{\text{out}}$. We have shown that the set S spans \overline{A} . The result follows. \square

21 The unipotent maps for an LR triple

Throughout this section the following notation is in effect. Let V denote a vector space over \mathbb{F} with dimension $d+1$. Let A, B, C denote an LR triple on V , with parameter array (44), idempotent data (48), trace data (50), and Toeplitz data (54). Using A, B, C we define three elements $\mathbb{A}, \mathbb{B}, \mathbb{C}$ in $\text{End}(V)$ called the unipotent maps. This name is motivated by Lemma 21.15 below.

Definition 21.1. Define

$$\mathbb{A} = \sum_{i=0}^d E_{d-i} E''_i, \quad \mathbb{B} = \sum_{i=0}^d E'_{d-i} E_i, \quad \mathbb{C} = \sum_{i=0}^d E''_{d-i} E'_i.$$

We call $\mathbb{A}, \mathbb{B}, \mathbb{C}$ the *unipotent maps* for A, B, C .

Lemma 21.2. Assume that A, B, C is trivial. Then $\mathbb{A} = \mathbb{B} = \mathbb{C} = I$.

Proof. For $d = 0$ we have $E_0 = E'_0 = E''_0 = I$. \square

Lemma 21.3. The maps $\mathbb{A}, \mathbb{B}, \mathbb{C}$ are invertible. Their inverses are

$$\mathbb{A}^{-1} = \sum_{i=0}^d E''_{d-i} E_i, \quad \mathbb{B}^{-1} = \sum_{i=0}^d E_{d-i} E'_i, \quad \mathbb{C}^{-1} = \sum_{i=0}^d E'_{d-i} E''_i.$$

Proof. Concerning \mathbb{A} and using Lemma 13.28,

$$\mathbb{A} \sum_{j=0}^d E''_{d-j} E_j = \sum_{j=0}^d E_j E''_{d-j} E_j = \sum_{j=0}^d E_j = I.$$

□

Lemma 21.4. *For $0 \leq i \leq d$,*

$$\mathbb{A} E''_i = E_{d-i} \mathbb{A}, \quad \mathbb{B} E_i = E'_{d-i} \mathbb{B}, \quad \mathbb{C} E'_i = E''_{d-i} \mathbb{C}.$$

Proof. These equations are verified by evaluating each side using Definition 21.1. □

Lemma 21.5. *For $0 \leq i \leq d$,*

$$\mathbb{A} E''_i V = E_{d-i} V, \quad \mathbb{B} E_i V = E'_{d-i} V, \quad \mathbb{C} E'_i V = E''_{d-i} V.$$

Proof. Use Lemmas 21.3, 21.4. □

Lemma 21.6. *The following (i)–(iii) hold:*

- (i) \mathbb{A} sends the (A, C) -decomposition of V to the (A, B) -decomposition of V ;
- (ii) \mathbb{B} sends the (B, A) -decomposition of V to the (B, C) -decomposition of V ;
- (iii) \mathbb{C} sends the (C, B) -decomposition of V to the (C, A) -decomposition of V .

Proof. This is a reformulation of Lemma 21.5. □

We now consider how the maps $\mathbb{A}, \mathbb{B}, \mathbb{C}$ act on the three flags (45).

Lemma 21.7. *The following (i)–(iii) hold:*

- (i) \mathbb{A} fixes $\{A^{d-i} V\}_{i=0}^d$ and sends $\{C^{d-i} V\}_{i=0}^d$ to $\{B^{d-i} V\}_{i=0}^d$;
- (ii) \mathbb{B} fixes $\{B^{d-i} V\}_{i=0}^d$ and sends $\{A^{d-i} V\}_{i=0}^d$ to $\{C^{d-i} V\}_{i=0}^d$;
- (iii) \mathbb{C} fixes $\{C^{d-i} V\}_{i=0}^d$ and sends $\{B^{d-i} V\}_{i=0}^d$ to $\{A^{d-i} V\}_{i=0}^d$.

Proof. By Lemmas 13.15, 21.6. □

We now consider how the maps $\mathbb{A}, \mathbb{B}, \mathbb{C}$ act on some bases for V .

Lemma 21.8. *The following (i)–(iii) hold:*

- (i) \mathbb{A} sends each (A, C) -basis of V to a compatible (A, B) -basis of V ;
- (ii) \mathbb{B} sends each (B, A) -basis of V to a compatible (B, C) -basis of V ;
- (iii) \mathbb{C} sends each (C, B) -basis of V to a compatible (C, A) -basis of V .

Proof. (i) Let $\{u_i\}_{i=0}^d$ denote an (A, C) -basis of V , and let $\{v_i\}_{i=0}^d$ denote a compatible (A, B) -basis of V . We have $Au_i = u_{i-1}$ for $1 \leq i \leq d$ and $Au_0 = 0$. Also $v_i = B^i v_0 / (\varphi_1 \cdots \varphi_i)$ for $0 \leq i \leq d$. Moreover $u_0 = v_0$. For $0 \leq i \leq d$,

$$\mathbb{A}u_i = E_i E''_{d-i} u_i = E_i u_i = \frac{A^{d-i} B^d A^i}{\varphi_1 \cdots \varphi_d} u_i = \frac{A^{d-i} B^d u_0}{\varphi_1 \cdots \varphi_d} = \frac{B^i u_0}{\varphi_1 \cdots \varphi_i} = v_i.$$

(ii), (iii) Similar to the proof of (i) above. □

Lemma 21.9. *The following (i)–(iii) hold:*

- (i) *the matrix T' represents \mathbb{A} with respect to each (A, C) -basis of V ;*
- (ii) *the matrix T'' represents \mathbb{B} with respect to each (B, A) -basis of V ;*
- (iii) *the matrix T represents \mathbb{C} with respect to each (C, B) -basis of V .*

Proof. By Definition 13.44 and Lemma 21.8. □

Recall the vectors η, η', η'' and $\tilde{\eta}, \tilde{\eta}', \tilde{\eta}''$ from (59), (60).

Lemma 21.10. *For $0 \leq i \leq d$,*

$$\begin{aligned} \mathbb{A}C^i \eta &= \frac{\varphi_d'' \cdots \varphi_{d-i+1}''}{\varphi_1 \cdots \varphi_i} B^i \eta, & \mathbb{A}^{-1} B^i \eta &= \frac{\varphi_1 \cdots \varphi_i}{\varphi_d'' \cdots \varphi_{d-i+1}''} C^i \eta, \\ \mathbb{B}A^i \eta' &= \frac{\varphi_d \cdots \varphi_{d-i+1}}{\varphi_1' \cdots \varphi_i'} C^i \eta', & \mathbb{B}^{-1} C^i \eta' &= \frac{\varphi_1' \cdots \varphi_i'}{\varphi_d \cdots \varphi_{d-i+1}} A^i \eta', \\ \mathbb{C}B^i \eta'' &= \frac{\varphi_d' \cdots \varphi_{d-i+1}'}{\varphi_1'' \cdots \varphi_i''} A^i \eta'', & \mathbb{C}^{-1} A^i \eta'' &= \frac{\varphi_1'' \cdots \varphi_i''}{\varphi_d' \cdots \varphi_{d-i+1}'} B^i \eta''. \end{aligned}$$

Proof. By Lemmas 13.54, 21.8. □

Lemma 21.11. *For $0 \leq i \leq d$,*

$$\begin{aligned} \mathbb{A}A^i \eta'' &= \frac{(\eta'', \tilde{\eta})}{(\eta', \tilde{\eta})} A^i \eta', & \mathbb{A}^{-1} A^i \eta' &= \frac{(\eta', \tilde{\eta})}{(\eta'', \tilde{\eta})} A^i \eta'', \\ \mathbb{B}B^i \eta &= \frac{(\eta, \tilde{\eta}')}{(\eta'', \tilde{\eta}')} B^i \eta'', & \mathbb{B}^{-1} B^i \eta'' &= \frac{(\eta'', \tilde{\eta}')}{(\eta, \tilde{\eta}')} B^i \eta, \\ \mathbb{C}C^i \eta' &= \frac{(\eta', \tilde{\eta}'')}{(\eta, \tilde{\eta}'')} C^i \eta, & \mathbb{C}^{-1} C^i \eta &= \frac{(\eta, \tilde{\eta}'')}{(\eta', \tilde{\eta}'')} C^i \eta'. \end{aligned}$$

Proof. Evaluate the displayed equations in Lemma 21.10 using Lemma 13.64, and simplify the results using Lemma 13.62 and Proposition 13.66. □

Lemma 21.12. *The following (i)–(iii) hold:*

- (i) *\mathbb{A} fixes η and sends η'' to $(\eta'', \tilde{\eta})/(\eta', \tilde{\eta})\eta'$;*
- (ii) *\mathbb{B} fixes η' and sends η to $(\eta, \tilde{\eta}')/(\eta'', \tilde{\eta}')\eta''$;*

(iii) \mathbb{C} fixes η'' and sends η' to $(\eta', \tilde{\eta}'')/(\eta, \tilde{\eta}'')\eta$.

Proof. Set $i = 0$ in Lemmas 21.10, 21.11. □

Proposition 21.13. *We have*

$$\mathbb{A} = \sum_{i=0}^d \alpha'_i A^i, \quad \mathbb{B} = \sum_{i=0}^d \alpha''_i B^i, \quad \mathbb{C} = \sum_{i=0}^d \alpha_i C^i. \quad (133)$$

Moreover

$$\mathbb{A}^{-1} = \sum_{i=0}^d \beta'_i A^i, \quad \mathbb{B}^{-1} = \sum_{i=0}^d \beta''_i B^i, \quad \mathbb{C}^{-1} = \sum_{i=0}^d \beta_i C^i. \quad (134)$$

Proof. We verify $\mathbb{A} = \sum_{i=0}^d \alpha'_i A^i$. Let $\{u_i\}_{i=0}^d$ denote an (A, C) -basis of V . Recall the matrix τ from Definition 3.49. By Proposition 13.39, τ represents A with respect to $\{u_i\}_{i=0}^d$. By Lemma 12.3, $T' = \sum_{i=0}^d \alpha'_i \tau^i$. So T' represents $\sum_{i=0}^d \alpha'_i A^i$ with respect to $\{u_i\}_{i=0}^d$. By Lemma 21.9(i), T' represents \mathbb{A} with respect to $\{u_i\}_{i=0}^d$. Therefore $\mathbb{A} = \sum_{i=0}^d \alpha'_i A^i$. The remaining assertions of the lemma are similarly verified. □

We emphasize one aspect of Proposition 21.13.

Corollary 21.14. *The element A (resp. B) (resp. C) commutes with \mathbb{A} (resp. \mathbb{B}) (resp. \mathbb{C}).*

An element $X \in \text{End}(V)$ is called *unipotent* whenever $X - I$ is nilpotent.

Lemma 21.15. *Each of \mathbb{A} , \mathbb{B} , \mathbb{C} is unipotent.*

Proof. The element $\mathbb{A} - I$ is nilpotent, since it is a linear combination of $\{A^i\}_{i=1}^d$ and A is nilpotent. Therefore \mathbb{A} is unipotent. The maps \mathbb{B} , \mathbb{C} are similarly shown to be unipotent. □

Definition 21.16. Call the sequence $\mathbb{A}, \mathbb{B}, \mathbb{C}$ the *unipotent data* for A, B, C .

Lemma 21.17. *Let α, β, γ denote nonzero scalars in \mathbb{F} . Then the LR triples A, B, C and $\alpha A, \beta B, \gamma C$ have the same unipotent data.*

Proof. By Lemma 13.22 and Definition 21.1. □

Lemma 21.18. *In the table below, we display some LR triples on V along with their unipotent data.*

LR triple	unipotent data
A, B, C	$\mathbb{A}, \mathbb{B}, \mathbb{C}$
B, C, A	$\mathbb{B}, \mathbb{C}, \mathbb{A}$
C, A, B	$\mathbb{C}, \mathbb{A}, \mathbb{B}$
C, B, A	$\mathbb{C}^{-1}, \mathbb{B}^{-1}, \mathbb{A}^{-1}$
A, C, B	$\mathbb{A}^{-1}, \mathbb{C}^{-1}, \mathbb{B}^{-1}$
B, A, C	$\mathbb{B}^{-1}, \mathbb{A}^{-1}, \mathbb{C}^{-1}$

Proof. Use Definition 21.1 and Lemmas 13.23, 21.3. \square

Lemma 21.19. *In the table below, we display some LR triples on V^* along with their unipotent data.*

LR triple	unipotent data
$\tilde{A}, \tilde{B}, \tilde{C}$	$\tilde{\mathbb{A}}^{-1}, \tilde{\mathbb{B}}^{-1}, \tilde{\mathbb{C}}^{-1}$
$\tilde{B}, \tilde{C}, \tilde{A}$	$\tilde{\mathbb{B}}^{-1}, \tilde{\mathbb{C}}^{-1}, \tilde{\mathbb{A}}^{-1}$
$\tilde{C}, \tilde{A}, \tilde{B}$	$\tilde{\mathbb{C}}^{-1}, \tilde{\mathbb{A}}^{-1}, \tilde{\mathbb{B}}^{-1}$
$\tilde{C}, \tilde{B}, \tilde{A}$	$\tilde{\mathbb{C}}, \tilde{\mathbb{B}}, \tilde{\mathbb{A}}$
$\tilde{A}, \tilde{C}, \tilde{B}$	$\tilde{\mathbb{A}}, \tilde{\mathbb{C}}, \tilde{\mathbb{B}}$
$\tilde{B}, \tilde{A}, \tilde{C}$	$\tilde{\mathbb{B}}, \tilde{\mathbb{A}}, \tilde{\mathbb{C}}$

Proof. Use Definition 21.1 and Lemmas 13.24, 21.3. Keep in mind that the adjoint map is an antiisomorphism. \square

Lemma 21.20. *Assume that A, B, C is bipartite. Then the projector J commutes with each of $\mathbb{A}, \mathbb{B}, \mathbb{C}$.*

Proof. By Definition 21.1, and since J commutes with each of E_i, E'_i, E''_i for $0 \leq i \leq d$. \square

Lemma 21.21. *Assume that A, B, C is bipartite. Then*

$$\begin{aligned} \mathbb{A}V_{\text{out}} &= V_{\text{out}}, & \mathbb{B}V_{\text{out}} &= V_{\text{out}}, & \mathbb{C}V_{\text{out}} &= V_{\text{out}}, \\ \mathbb{A}V_{\text{in}} &= V_{\text{in}}, & \mathbb{B}V_{\text{in}} &= V_{\text{in}}, & \mathbb{C}V_{\text{in}} &= V_{\text{in}}. \end{aligned}$$

Proof. By Lemma 16.12, Definition 21.1, and since d is even, we find that V_{out} and V_{in} are invariant under each of $\mathbb{A}, \mathbb{B}, \mathbb{C}$. By Lemma 21.3 the maps $\mathbb{A}, \mathbb{B}, \mathbb{C}$ are invertible. The result follows. \square

The next two lemmas follow from the construction.

Lemma 21.22. *Assume that A, B, C is bipartite, so that A^2, B^2, C^2 act on V_{out} as an LR triple. The unipotent data for this triple is given by the actions of $\mathbb{A}, \mathbb{B}, \mathbb{C}$ on V_{out} .*

Lemma 21.23. *Assume that A, B, C is bipartite and nontrivial, so that A^2, B^2, C^2 act on V_{in} as an LR triple. The unipotent data for this triple is given by the actions of $\mathbb{A}, \mathbb{B}, \mathbb{C}$ on V_{in} .*

Lemma 21.24. *Assume that A, B, C is bipartite. Let*

$$\alpha_{\text{out}}, \quad \alpha_{\text{in}}, \quad \beta_{\text{out}}, \quad \beta_{\text{in}}, \quad \gamma_{\text{out}}, \quad \gamma_{\text{in}}$$

denote nonzero scalars in \mathbb{F} , so that the sequence

$$\alpha_{\text{out}}A_{\text{out}} + \alpha_{\text{in}}A_{\text{in}}, \quad \beta_{\text{out}}B_{\text{out}} + \beta_{\text{in}}B_{\text{in}}, \quad \gamma_{\text{out}}C_{\text{out}} + \gamma_{\text{in}}C_{\text{in}}$$

is a bipartite LR triple on V . This LR triple has the same unipotent data as A, B, C .

Proof. By Lemma 16.33(ii) and Definition 21.1. \square

22 The rotators for an LR triple

Throughout this section the following notation is in effect. Let V denote a vector space over \mathbb{F} with dimension $d + 1$. Let A, B, C denote an LR triple on V , with parameter array (44), idempotent data (48), trace data (50), and Toeplitz data (54). We discuss a type of element in $\text{End}(V)$ called a rotator.

Definition 22.1. By a *rotator* for A, B, C we mean an element $R \in \text{End}(V)$ such that for $0 \leq i \leq d$,

$$E_i R = R E'_i, \quad E'_i R = R E''_i, \quad E''_i R = R E_i. \quad (135)$$

Lemma 22.2. For $R \in \text{End}(V)$ the following are equivalent:

- (i) R is a rotator for A, B, C ;
- (ii) for $0 \leq i \leq d$,

$$R E'_i V \subseteq E_i V, \quad R E''_i V \subseteq E'_i V, \quad R E_i V \subseteq E''_i V.$$

Proof. By Definition 22.1 and linear algebra. □

Lemma 22.3. Assume that A, B, C is trivial. Then each element of $\text{End}(V)$ is a rotator for A, B, C .

Proof. For $d = 0$ we have $E_0 = E'_0 = E''_0 = I$. □

Lemma 22.4. Let R denote a rotator for A, B, C . Then

$$\mathbb{A}R = R\mathbb{B}, \quad \mathbb{B}R = R\mathbb{C}, \quad \mathbb{C}R = R\mathbb{A}. \quad (136)$$

Proof. Use Definitions 21.1, 22.1. □

Definition 22.5. Let \mathcal{R} denote the set of rotators for A, B, C . Note that \mathcal{R} is a subspace of the \mathbb{F} -vector space $\text{End}(V)$. We call \mathcal{R} the *rotator space* for A, B, C .

Definition 22.6. Assume that A, B, C is trivial. Then the identity I of $\text{End}(V) = \mathcal{R}$ is a basis for \mathcal{R} . We call I the *standard rotator* for A, B, C .

Assume for the moment that A, B, C is nontrivial. We are going to show that \mathcal{R} has dimension 1 (if A, B, C is nonbipartite) and 2 (if A, B, C is bipartite). In each case, we will display an explicit basis for \mathcal{R} . We now obtain some results that will be used to construct these bases.

Lemma 22.7. The following (i)–(iii) hold.

- (i) $\mathbb{B}^{-1}C\mathbb{B}$ is zero on E_0V . Moreover for $1 \leq i \leq d$ and on E_iV ,

$$\mathbb{B}^{-1}C\mathbb{B} = \frac{\varphi'_{d-i+1}}{\varphi_i} A.$$

(ii) $\mathbb{C}^{-1}AC$ is zero on E'_0V . Moreover for $1 \leq i \leq d$ and on E'_iV ,

$$\mathbb{C}^{-1}AC = \frac{\varphi''_{d-i+1}}{\varphi'_i}B.$$

(iii) $\mathbb{A}^{-1}BA$ is zero on E''_0V . Moreover for $1 \leq i \leq d$ and on E''_iV ,

$$\mathbb{A}^{-1}BA = \frac{\varphi_{d-i+1}}{\varphi''_i}C.$$

Proof. (i) The vector $A^{d-i}\eta'$ is a basis for E_iV . Apply $\mathbb{B}^{-1}C\mathbb{B}$ to this vector and evaluate the result using Lemma 21.10 (middle row).

(ii), (iii) Similar to the proof of (i) above. \square

Lemma 22.8. *The following (i)–(iii) hold.*

(i) $\mathbb{A}C\mathbb{A}^{-1}$ is zero on E_dV . Moreover for $0 \leq i \leq d-1$ and on E_iV ,

$$\mathbb{A}C\mathbb{A}^{-1} = \frac{\varphi''_{d-i}}{\varphi_{i+1}}B.$$

(ii) $\mathbb{B}A\mathbb{B}^{-1}$ is zero on E'_dV . Moreover for $0 \leq i \leq d-1$ and on E'_iV ,

$$\mathbb{B}A\mathbb{B}^{-1} = \frac{\varphi_{d-i}}{\varphi'_{i+1}}C.$$

(iii) $\mathbb{C}B\mathbb{C}^{-1}$ is zero on E''_dV . Moreover for $0 \leq i \leq d-1$ and on E''_iV ,

$$\mathbb{C}B\mathbb{C}^{-1} = \frac{\varphi'_{d-i}}{\varphi''_{i+1}}A.$$

Proof. (i) The vector $B^i\eta$ is a basis for E_iV . Apply $\mathbb{A}C\mathbb{A}^{-1}$ to this vector and evaluate the result using Lemma 21.10 (top row).

(ii), (iii) Similar to the proof of (i) above. \square

Lemma 22.9. *The following (i)–(iii) hold:*

(i) the (A, B) -decomposition of V is lowered by $\mathbb{B}^{-1}C\mathbb{B}$ and raised by $\mathbb{A}C\mathbb{A}^{-1}$;

(ii) the (B, C) -decomposition of V is lowered by $\mathbb{C}^{-1}AC$ and raised by $\mathbb{B}A\mathbb{B}^{-1}$;

(iii) the (C, A) -decomposition of V is lowered by $\mathbb{A}^{-1}BA$ and raised by $\mathbb{C}B\mathbb{C}^{-1}$.

Proof. (i) The sequence $\{E_iV\}_{i=0}^d$ is the (A, B) -decomposition of V . This decomposition is lowered by A and raised by B . The result follows in view of Lemmas 22.7(i), 22.8(i). \square

Lemma 22.10. *We have*

$$\begin{aligned}\mathbb{B}^{-1}C\mathbb{B}\left(\sum_{i=0}^d\frac{\varphi_1\cdots\varphi_i}{\varphi'_d\cdots\varphi'_{d-i+1}}E_i\right) &= \left(\sum_{i=0}^d\frac{\varphi_1\cdots\varphi_i}{\varphi'_d\cdots\varphi'_{d-i+1}}E_i\right)A, \\ \mathbb{C}^{-1}A\mathbb{C}\left(\sum_{i=0}^d\frac{\varphi'_1\cdots\varphi'_i}{\varphi''_d\cdots\varphi''_{d-i+1}}E'_i\right) &= \left(\sum_{i=0}^d\frac{\varphi'_1\cdots\varphi'_i}{\varphi''_d\cdots\varphi''_{d-i+1}}E'_i\right)B, \\ \mathbb{A}^{-1}B\mathbb{A}\left(\sum_{i=0}^d\frac{\varphi''_1\cdots\varphi''_i}{\varphi_d\cdots\varphi_{d-i+1}}E''_i\right) &= \left(\sum_{i=0}^d\frac{\varphi''_1\cdots\varphi''_i}{\varphi_d\cdots\varphi_{d-i+1}}E''_i\right)C\end{aligned}$$

and also

$$\begin{aligned}\left(\sum_{i=0}^d\frac{\varphi_1\cdots\varphi_i}{\varphi''_d\cdots\varphi''_{d-i+1}}E_i\right)\mathbb{A}C\mathbb{A}^{-1} &= B\left(\sum_{i=0}^d\frac{\varphi_1\cdots\varphi_i}{\varphi''_d\cdots\varphi''_{d-i+1}}E_i\right), \\ \left(\sum_{i=0}^d\frac{\varphi'_1\cdots\varphi'_i}{\varphi_d\cdots\varphi_{d-i+1}}E'_i\right)\mathbb{B}A\mathbb{B}^{-1} &= C\left(\sum_{i=0}^d\frac{\varphi'_1\cdots\varphi'_i}{\varphi_d\cdots\varphi_{d-i+1}}E'_i\right), \\ \left(\sum_{i=0}^d\frac{\varphi''_1\cdots\varphi''_i}{\varphi'_d\cdots\varphi'_{d-i+1}}E''_i\right)\mathbb{C}B\mathbb{C}^{-1} &= A\left(\sum_{i=0}^d\frac{\varphi''_1\cdots\varphi''_i}{\varphi'_d\cdots\varphi'_{d-i+1}}E''_i\right).\end{aligned}$$

Proof. To verify the first (resp. fourth) displayed equation in the lemma statement, for $0 \leq i \leq d$ apply each side to E_iV , and evaluate the result using Lemma 22.7(i) (resp. Lemma 22.8(i)). The remaining equations are similarly verified. \square

For the next few results, it is convenient to assume that A, B, C is equitable. Shortly we will return to the general case.

Proposition 22.11. *Assume that A, B, C is equitable. Then*

$$\mathbb{C}\left(\sum_{i=0}^d\frac{\varphi'_1\cdots\varphi'_i}{\varphi''_d\cdots\varphi''_{d-i+1}}E'_i\right)\mathbb{B} = \mathbb{A}\left(\sum_{i=0}^d\frac{\varphi''_1\cdots\varphi''_i}{\varphi'_d\cdots\varphi'_{d-i+1}}E''_i\right)\mathbb{C}, \quad (137)$$

$$\mathbb{A}\left(\sum_{i=0}^d\frac{\varphi''_1\cdots\varphi''_i}{\varphi_d\cdots\varphi_{d-i+1}}E''_i\right)\mathbb{C} = \mathbb{B}\left(\sum_{i=0}^d\frac{\varphi_1\cdots\varphi_i}{\varphi''_d\cdots\varphi''_{d-i+1}}E_i\right)\mathbb{A}, \quad (138)$$

$$\mathbb{B}\left(\sum_{i=0}^d\frac{\varphi_1\cdots\varphi_i}{\varphi'_d\cdots\varphi'_{d-i+1}}E_i\right)\mathbb{A} = \mathbb{C}\left(\sum_{i=0}^d\frac{\varphi'_1\cdots\varphi'_i}{\varphi_d\cdots\varphi_{d-i+1}}E'_i\right)\mathbb{B}. \quad (139)$$

Proof. We prove (137). Define

$$X = \mathbb{C}\left(\sum_{i=0}^d\frac{\varphi'_1\cdots\varphi'_i}{\varphi''_d\cdots\varphi''_{d-i+1}}E'_i\right). \quad (140)$$

We claim that

$$X = \left(\sum_{i=0}^d\frac{\varphi''_1\cdots\varphi''_i}{\varphi'_d\cdots\varphi'_{d-i+1}}E''_i\right)\mathbb{C}. \quad (141)$$

To verify (141), evaluate the right-hand side of (140) using Lemma 21.4, and simplify the result using Lemma 17.21. The claim is proven. By (141) and the last displayed equation in Lemma 22.10, $XB = AX$. So $XB^i = A^iX$ for $0 \leq i \leq d$. By Definition 17.1 and line (133), $\mathbb{A} = \sum_{i=0}^d \alpha_i A^i$ and $\mathbb{B} = \sum_{i=0}^d \alpha_i B^i$. By these comments $X\mathbb{B} = \mathbb{A}X$. In this equation evaluate the X on the left and right using (140) and (141), respectively. This yields (137). The equations (138), (139) are similarly obtained. \square

Definition 22.12. Assume that A, B, C is equitable. Let $\Omega, \Omega', \Omega''$ denote the common values of (137), (138), (139) respectively.

Lemma 22.13. Assume that A, B, C is trivial. Then $\Omega = \Omega' = \Omega'' = I$.

Proof. By Lemma 21.2, Definition 22.12, and since $E_0 = E'_0 = E''_0 = I$. \square

Lemma 22.14. Assume that A, B, C is equitable. Then for $0 \leq i \leq d$,

$$E_i \Omega = \Omega E'_i, \quad E'_i \Omega' = \Omega' E''_i, \quad E''_i \Omega'' = \Omega'' E_i.$$

Proof. To verify $E_i \Omega = \Omega E'_i$, eliminate Ω using the formula on the right in (137), and evaluate the result using Lemma 21.4. The remaining equations are similarly verified. \square

Lemma 22.15. Assume that A, B, C is equitable. Then

$$A\Omega = \Omega B, \quad B\Omega' = \Omega' C, \quad C\Omega'' = \Omega'' A. \quad (142)$$

Proof. To verify $A\Omega = \Omega B$, eliminate Ω using the formula on the right in (137), and evaluate the result using Corollary 21.14 and the last displayed equation in Lemma 22.10. The remaining equations in (142) are similarly verified. \square

Lemma 22.16. Assume that A, B, C is equitable. Then the elements $\Omega, \Omega', \Omega''$ are invertible. Moreover

$$\begin{aligned} \Omega^{-1} &= \mathbb{B}^{-1} \left(\sum_{i=0}^d \frac{\varphi''_d \cdots \varphi''_{d-i+1}}{\varphi'_1 \cdots \varphi'_i} E'_i \right) \mathbb{C}^{-1} = \mathbb{C}^{-1} \left(\sum_{i=0}^d \frac{\varphi'_d \cdots \varphi'_{d-i+1}}{\varphi''_1 \cdots \varphi''_i} E''_i \right) \mathbb{A}^{-1}, \\ (\Omega')^{-1} &= \mathbb{C}^{-1} \left(\sum_{i=0}^d \frac{\varphi_d \cdots \varphi_{d-i+1}}{\varphi''_1 \cdots \varphi''_i} E''_i \right) \mathbb{A}^{-1} = \mathbb{A}^{-1} \left(\sum_{i=0}^d \frac{\varphi''_d \cdots \varphi''_{d-i+1}}{\varphi_1 \cdots \varphi_i} E_i \right) \mathbb{B}^{-1}, \\ (\Omega'')^{-1} &= \mathbb{A}^{-1} \left(\sum_{i=0}^d \frac{\varphi'_d \cdots \varphi'_{d-i+1}}{\varphi_1 \cdots \varphi_i} E_i \right) \mathbb{B}^{-1} = \mathbb{B}^{-1} \left(\sum_{i=0}^d \frac{\varphi_d \cdots \varphi_{d-i+1}}{\varphi'_1 \cdots \varphi'_i} E'_i \right) \mathbb{C}^{-1}. \end{aligned}$$

Proof. Use Proposition 22.11 and Definition 22.12. \square

Lemma 22.17. Assume that A, B, C is equitable and nonbipartite. Then $\Omega = \Omega' = \Omega''$, and this common value is equal to

$$\begin{aligned} \mathbb{B} \left(\sum_{i=0}^d \frac{\varphi_1 \cdots \varphi_i}{\varphi_d \cdots \varphi_{d-i+1}} E_i \right) \mathbb{A} &= \mathbb{C} \left(\sum_{i=0}^d \frac{\varphi_1 \cdots \varphi_i}{\varphi_d \cdots \varphi_{d-i+1}} E'_i \right) \mathbb{B} \\ &= \mathbb{A} \left(\sum_{i=0}^d \frac{\varphi_1 \cdots \varphi_i}{\varphi_d \cdots \varphi_{d-i+1}} E''_i \right) \mathbb{C}. \end{aligned}$$

Proof. By Lemma 17.6(i) along with (137)–(139) and Definition 22.12. \square

For the past few results we assumed that A, B, C is equitable. We now drop the equitable assumption and return to the general case.

Theorem 22.18. *Assume that A, B, C is nonbipartite. Then the following (i)–(v) hold.*

(i) *We have*

$$\begin{aligned} \mathbb{B} \left(\sum_{i=0}^d \frac{\varphi_1 \cdots \varphi_i}{\varphi_d \cdots \varphi_{d-i+1}} E_i \right) \mathbb{A} &= \mathbb{C} \left(\sum_{i=0}^d \frac{\varphi_1 \cdots \varphi_i}{\varphi_d \cdots \varphi_{d-i+1}} E'_i \right) \mathbb{B} \\ &= \mathbb{A} \left(\sum_{i=0}^d \frac{\varphi_1 \cdots \varphi_i}{\varphi_d \cdots \varphi_{d-i+1}} E''_i \right) \mathbb{C}. \end{aligned}$$

Denote this common value by Ω .

(ii) *Ω is invertible, and Ω^{-1} is equal to*

$$\begin{aligned} \mathbb{A}^{-1} \left(\sum_{i=0}^d \frac{\varphi_d \cdots \varphi_{d-i+1}}{\varphi_1 \cdots \varphi_i} E_i \right) \mathbb{B}^{-1} &= \mathbb{B}^{-1} \left(\sum_{i=0}^d \frac{\varphi_d \cdots \varphi_{d-i+1}}{\varphi_1 \cdots \varphi_i} E'_i \right) \mathbb{C}^{-1} \\ &= \mathbb{C}^{-1} \left(\sum_{i=0}^d \frac{\varphi_d \cdots \varphi_{d-i+1}}{\varphi_1 \cdots \varphi_i} E''_i \right) \mathbb{A}^{-1}. \end{aligned}$$

(iii) *For $0 \leq i \leq d$,*

$$E_i \Omega = \Omega E'_i, \quad E'_i \Omega = \Omega E''_i, \quad E''_i \Omega = \Omega E_i.$$

(iv) *We have*

$$\alpha'_1 A \Omega = \alpha''_1 \Omega B, \quad \alpha''_1 B \Omega = \alpha_1 \Omega C, \quad \alpha_1 C \Omega = \alpha'_1 \Omega A.$$

(v) *For A, B, C equitable,*

$$A \Omega = \Omega B, \quad B \Omega = \Omega C, \quad C \Omega = \Omega A.$$

Proof. Apply Lemmas 22.14–22.17 to the equitable LR triple $\alpha'_1 A, \alpha''_1 B, \alpha_1 C$ and use Lemmas 13.7, 13.22, 21.17. \square

Proposition 22.19. *Assume that A, B, C is nonbipartite. Then Ω is a rotator for A, B, C .*

Proof. By Definition 22.1 and Theorem 22.18(iii). \square

Lemma 22.20. *Assume that A, B, C is nonbipartite. Then for $0 \leq i \leq d$,*

$$\Omega E'_i V = E_i V, \quad \Omega E''_i V = E'_i V, \quad \Omega E_i V = E''_i V.$$

Proof. By Proposition 22.19 the map Ω is a rotator for A, B, C . The result follows by Lemma 22.2, and since Ω is invertible by Theorem 22.18(ii). \square

Recall the rotator space \mathcal{R} from Definition 22.5.

Proposition 22.21. *Assume that A, B, C is nonbipartite. Then Ω is a basis for the \mathbb{F} -vector space \mathcal{R} .*

Proof. We have $\Omega \in \mathcal{R}$ by Proposition 22.19. The map Ω is invertible by Theorem 22.18(ii), so of course $\Omega \neq 0$. We show that Ω spans \mathcal{R} . Let $R \in \mathcal{R}$. By assumption R and Ω are rotators for A, B, C . So by Definition 22.1, $\Omega^{-1}R$ commutes with each of E_i, E'_i, E''_i for $0 \leq i \leq d$. Now by Definition 19.1, $\Omega^{-1}R$ is an idempotent centralizer for A, B, C . By Definition 19.4 and Proposition 19.5(ii), there exists $\zeta \in \mathbb{F}$ such that $\Omega^{-1}R = \zeta I$. Therefore $R = \zeta\Omega$. We have shown that Ω spans \mathcal{R} . The result follows. \square

Definition 22.22. Assume that A, B, C is nonbipartite. By the *standard rotator* for A, B, C we mean the map Ω from Theorem 22.18 and Proposition 22.21.

Lemma 22.23. *Assume that A, B, C is nonbipartite. Then Ω sends*

$$\eta \rightarrow \frac{(\eta, \tilde{\eta}')}{(\eta'', \tilde{\eta}')}\eta'', \quad \eta' \rightarrow \frac{(\eta', \tilde{\eta}'')}{(\eta, \tilde{\eta}'')}\eta, \quad \eta'' \rightarrow \frac{(\eta'', \tilde{\eta})}{(\eta', \tilde{\eta})}\eta'.$$

Proof. Use Lemma 21.12 and the formulae for Ω given in Theorem 22.18(i). \square

Proposition 22.24. *Assume that A, B, C is nonbipartite. Then $\Omega^3 = \theta I$, where θ is from Definition 13.71.*

Proof. By Theorem 22.18(iii), Ω^3 commutes with each of E_i, E'_i, E''_i for $0 \leq i \leq d$. By Definition 19.1, Ω^3 is an idempotent centralizer for A, B, C . By Definition 19.4 and Proposition 19.5(ii), there exists $\zeta \in \mathbb{F}$ such that $\Omega^3 = \zeta I$. Considering the action of Ω^3 on η and using Lemma 22.23, we obtain $\zeta = \theta$. \square

Lemma 22.25. *Assume that A, B, C is nonbipartite. Let α, β, γ denote nonzero scalars in \mathbb{F} . Then the LR triples A, B, C and $\alpha A, \beta B, \gamma C$ have the same standard rotator.*

Proof. Use Lemmas 13.7, 13.22, 21.17 and any formula for Ω given in Theorem 22.18(i). \square

Lemma 22.26. *Assume that A, B, C is nonbipartite.*

(i) *The following LR triples have the same standard rotator:*

$$A, B, C \quad B, C, A \quad C, A, B.$$

(ii) *The following LR triples have the same standard rotator:*

$$C, B, A \quad A, C, B \quad B, A, C.$$

(iii) *The standard rotators in (i), (ii) above are inverses.*

Proof. By Lemmas 17.10, 22.25 we may assume without loss that A, B, C is equitable. Now use Lemmas 13.9, 13.23, 17.6(i), 21.18 and the formulae for Ω, Ω^{-1} given in Theorem 22.18(i),(ii). \square

Recall from Lemma 13.13 the LR triple $\tilde{A}, \tilde{B}, \tilde{C}$ on the dual space V^* .

Lemma 22.27. *Assume that A, B, C is nonbipartite. Then the following are inverse:*

- (i) *the adjoint of the standard rotator for A, B, C ;*
- (ii) *the standard rotator for $\tilde{A}, \tilde{B}, \tilde{C}$.*

Proof. Use Lemmas 13.13, 13.24, 21.19 and any formula for Ω given in Theorem 22.18(i). \square

Proposition 22.28. *Assume that A, B, C is nonbipartite. Let R denote a nonzero rotator for A, B, C . Then the following (i)–(iv) hold.*

- (i) *R is invertible.*
- (ii) *We have*

$$\alpha'_1 AR = \alpha''_1 RB, \quad \alpha''_1 BR = \alpha_1 RC, \quad \alpha_1 CR = \alpha'_1 RA.$$

- (iii) *We have*

$$\overline{AR} = R\overline{B}, \quad \overline{BR} = R\overline{C}, \quad \overline{CR} = R\overline{A}.$$

- (iv) *For A, B, C equitable,*

$$AR = RB, \quad BR = RC, \quad CR = RA.$$

Proof. By Proposition 22.21 there exists $0 \neq \zeta \in \mathbb{F}$ such that $R = \zeta\Omega$. The results follow in view of Theorems 20.3(ii), 22.18. \square

We now turn our attention to the case in which A, B, C is bipartite and nontrivial.

Lemma 22.29. *Assume that A, B, C is bipartite and nontrivial. Let R denote a rotator for A, B, C . Then the following (i)–(iii) hold:*

- (i) *$RV_{\text{out}} \subseteq V_{\text{out}}$ and $RV_{\text{in}} \subseteq V_{\text{in}}$;*
- (ii) *$RJ = JR$;*
- (iii) *RJ is a rotator for A, B, C .*

Proof. (i) By Lemmas 16.12, 22.2.

(ii) The map J acts on V_{out} as the identity, and on V_{in} as zero. The result follows from this and (i) above.

(iii) By Definition 22.1, and since J is an idempotent centralizer for A, B, C by Proposition 19.5(iii). \square

Definition 22.30. Assume that A, B, C is bipartite and nontrivial. Let R denote a rotator for A, B, C . Then R is called *outer* (resp. *inner*) whenever R is zero on V_{in} (resp. V_{out}). Let \mathcal{R}_{out} (resp. \mathcal{R}_{in}) denote the set of outer (resp. inner) rotators for A, B, C . Note that \mathcal{R}_{out} and \mathcal{R}_{in} are subspaces of the \mathbb{F} -vector space \mathcal{R} .

Definition 22.31. Assume that A, B, C is bipartite and nontrivial. Define elements $\Omega_{\text{out}}, \Omega_{\text{in}}$ in $\text{End}(V)$ as follows. Recall by Lemmas 16.17, 16.19(ii) that A^2, B^2, C^2 acts on V_{out} as a nonbipartite LR triple. The map Ω_{out} acts on V_{out} as the standard rotator for this LR triple. The map Ω_{out} acts on V_{in} as zero. The map Ω_{in} acts on V_{out} as zero. Recall by Lemmas 16.18, 16.20(i),(ii) that A^2, B^2, C^2 acts on V_{in} as an LR triple that is nonbipartite or trivial. The map Ω_{in} acts on V_{in} as the standard rotator for this LR triple.

Lemma 22.32. *With reference to Definition 22.31,*

$$\Omega_{\text{out}}V_{\text{out}} = V_{\text{out}}, \quad \Omega_{\text{out}}V_{\text{in}} = 0, \quad \Omega_{\text{in}}V_{\text{out}} = 0, \quad \Omega_{\text{in}}V_{\text{in}} = V_{\text{in}}.$$

Proof. By Definition 22.31 and the construction. \square

Proposition 22.33. *Assume that A, B, C is bipartite and nontrivial. Then the following (i)–(iii) hold:*

- (i) *the sum $\mathcal{R} = \mathcal{R}_{\text{out}} + \mathcal{R}_{\text{in}}$ is direct;*
- (ii) *Ω_{out} is a basis for \mathcal{R}_{out} ;*
- (iii) *Ω_{in} is a basis for \mathcal{R}_{in} .*

Proof. By Definitions 22.30, 22.31 we find $\Omega_{\text{out}} \in \mathcal{R}_{\text{out}}$ and $\Omega_{\text{in}} \in \mathcal{R}_{\text{in}}$. We mentioned in Definition 22.31 that A^2, B^2, C^2 acts on V_{out} as a nonbipartite LR triple. Denote the corresponding rotator subspace and standard rotator by \mathcal{R}^{out} and Ω^{out} , respectively. By construction \mathcal{R}^{out} is a subspace of $\text{End}(V_{\text{out}})$. By Proposition 22.21, Ω^{out} is a basis for \mathcal{R}^{out} . For $R \in \mathcal{R}$ the restriction $R|_{V_{\text{out}}}$ is contained in \mathcal{R}^{out} , and the map $\mathcal{R} \rightarrow \mathcal{R}^{\text{out}}, R \mapsto R|_{V_{\text{out}}}$ is \mathbb{F} -linear. This map has kernel \mathcal{R}_{in} . This map sends $\Omega_{\text{out}} \mapsto \Omega^{\text{out}}$ and is therefore surjective. By these comments Ω_{out} forms a basis for a complement of \mathcal{R}_{in} in \mathcal{R} . Similarly Ω_{in} forms a basis for a complement of \mathcal{R}_{out} in \mathcal{R} . Note that $\mathcal{R}_{\text{out}} \cap \mathcal{R}_{\text{in}} = 0$ by Definition 22.30 and since the sum $V = V_{\text{out}} + V_{\text{in}}$ is direct. The result follows. \square

Definition 22.34. With reference to Definition 22.31 and Proposition 22.33, we call Ω_{out} (resp. Ω_{in}) the *standard outer rotator* (resp. *standard inner rotator*) for A, B, C .

We now describe Ω_{out} and Ω_{in} in more detail.

Theorem 22.35. *Assume that A, B, C is bipartite and nontrivial. Then the following (i)–(v) hold.*

- (i) *We have*

$$\begin{aligned} \Omega_{\text{out}} &= \mathbb{B} \left(\sum_{j=0}^{d/2} \frac{\varphi_1 \varphi_2 \cdots \varphi_{2j}}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-2j+1}} E_{2j} \right) \mathbb{A} = \mathbb{C} \left(\sum_{j=0}^{d/2} \frac{\varphi_1 \varphi_2 \cdots \varphi_{2j}}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-2j+1}} E'_{2j} \right) \mathbb{B} \\ &= \mathbb{A} \left(\sum_{j=0}^{d/2} \frac{\varphi_1 \varphi_2 \cdots \varphi_{2j}}{\varphi_d \varphi_{d-1} \cdots \varphi_{d-2j+1}} E''_{2j} \right) \mathbb{C}. \end{aligned}$$

(ii) For $0 \leq i \leq d$,

$$E_i \Omega_{\text{out}} = \Omega_{\text{out}} E'_i, \quad E'_i \Omega_{\text{out}} = \Omega_{\text{out}} E''_i, \quad E''_i \Omega_{\text{out}} = \Omega_{\text{out}} E_i. \quad (143)$$

(iii) Referring to (143), if i is odd then for each equation both sides are zero.

(iv) We have

$$\alpha'_2 A^2 \Omega_{\text{out}} = \alpha''_2 \Omega_{\text{out}} B^2, \quad \alpha''_2 B^2 \Omega_{\text{out}} = \alpha_2 \Omega_{\text{out}} C^2, \quad \alpha_2 C^2 \Omega_{\text{out}} = \alpha'_2 \Omega_{\text{out}} A^2.$$

(v) For A, B, C equitable,

$$A^2 \Omega_{\text{out}} = \Omega_{\text{out}} B^2, \quad B^2 \Omega_{\text{out}} = \Omega_{\text{out}} C^2, \quad C^2 \Omega_{\text{out}} = \Omega_{\text{out}} A^2.$$

Proof. Apply Theorem 22.18 to the LR triple in Lemma 16.17, and evaluate the result using Lemmas 21.22, 22.32. \square

Theorem 22.36. Assume that A, B, C is bipartite and nontrivial. Then the following (i)–(v) hold.

(i) We have

$$\begin{aligned} \Omega_{\text{in}} &= \mathbb{B} \left(\sum_{j=0}^{d/2-1} \frac{\varphi_2 \varphi_3 \cdots \varphi_{2j+1}}{\varphi_{d-1} \varphi_{d-2} \cdots \varphi_{d-2j}} E_{2j+1} \right) \mathbb{A} = \mathbb{C} \left(\sum_{j=0}^{d/2-1} \frac{\varphi_2 \varphi_3 \cdots \varphi_{2j+1}}{\varphi_{d-1} \varphi_{d-2} \cdots \varphi_{d-2j}} E'_{2j+1} \right) \mathbb{B} \\ &= \mathbb{A} \left(\sum_{j=0}^{d/2-1} \frac{\varphi_2 \varphi_3 \cdots \varphi_{2j+1}}{\varphi_{d-1} \varphi_{d-2} \cdots \varphi_{d-2j}} E''_{2j+1} \right) \mathbb{C}. \end{aligned}$$

(ii) For $0 \leq i \leq d$,

$$E_i \Omega_{\text{in}} = \Omega_{\text{in}} E'_i, \quad E'_i \Omega_{\text{in}} = \Omega_{\text{in}} E''_i, \quad E''_i \Omega_{\text{in}} = \Omega_{\text{in}} E_i. \quad (144)$$

(iii) Referring to (144), if i is even then for each equation both sides are zero.

(iv) We have

$$\alpha'_2 A^2 \Omega_{\text{in}} = \alpha''_2 \Omega_{\text{in}} B^2, \quad \alpha''_2 B^2 \Omega_{\text{in}} = \alpha_2 \Omega_{\text{in}} C^2, \quad \alpha_2 C^2 \Omega_{\text{in}} = \alpha'_2 \Omega_{\text{in}} A^2.$$

(v) For A, B, C equitable,

$$A^2 \Omega_{\text{in}} = \Omega_{\text{in}} B^2, \quad B^2 \Omega_{\text{in}} = \Omega_{\text{in}} C^2, \quad C^2 \Omega_{\text{in}} = \Omega_{\text{in}} A^2.$$

Proof. Apply Theorem 22.18 to the LR triple in Lemma 16.18, and evaluate the result using Lemmas 21.23, 22.32. \square

Recall the maps $\Omega, \Omega', \Omega''$ from Definition 22.12.

Proposition 22.37. *Assume that A, B, C is equitable, bipartite, and nontrivial. Then*

$$\Omega = \Omega_{\text{out}} + \rho_0 \Omega_{\text{in}}, \quad \Omega' = \Omega_{\text{out}} + \rho'_0 \Omega_{\text{in}}, \quad \Omega'' = \Omega_{\text{out}} + \rho''_0 \Omega_{\text{in}}.$$

Proof. Compare the formulae for $\Omega, \Omega', \Omega''$ given in Proposition 22.11, with the formulae for $\Omega_{\text{out}}, \Omega_{\text{in}}$ given in Theorems 22.35(i), 22.36(i). The result follows in view of Lemma 17.13 and (129). \square

Lemma 22.38. *Assume that A, B, C is equitable, bipartite, and nontrivial. Then $\Omega, \Omega', \Omega''$ are rotators for A, B, C .*

Proof. By Propositions 22.33, 22.37. \square

Proposition 22.39. *Assume that A, B, C is bipartite and nontrivial. Then*

$$\begin{aligned} \varphi'_d A \Omega_{\text{out}} &= \varphi''_1 \Omega_{\text{in}} B, & \varphi''_d B \Omega_{\text{out}} &= \varphi_1 \Omega_{\text{in}} C, & \varphi_d C \Omega_{\text{out}} &= \varphi'_1 \Omega_{\text{in}} A, \\ \varphi'_1 A \Omega_{\text{in}} &= \varphi''_d \Omega_{\text{out}} B, & \varphi''_1 B \Omega_{\text{in}} &= \varphi_d \Omega_{\text{out}} C, & \varphi_1 C \Omega_{\text{in}} &= \varphi'_d \Omega_{\text{out}} A. \end{aligned}$$

Moreover for A, B, C equitable,

$$\begin{aligned} A \Omega_{\text{out}} &= \rho_0 \Omega_{\text{in}} B, & B \Omega_{\text{out}} &= \rho'_0 \Omega_{\text{in}} C, & C \Omega_{\text{out}} &= \rho''_0 \Omega_{\text{in}} A, \\ \rho_0 A \Omega_{\text{in}} &= \Omega_{\text{out}} B, & \rho'_0 B \Omega_{\text{in}} &= \Omega_{\text{out}} C, & \rho''_0 C \Omega_{\text{in}} &= \Omega_{\text{out}} A. \end{aligned}$$

Proof. First assume that A, B, C is equitable. To obtain the result under this assumption, evaluate (142) using Proposition 22.37, Lemmas 16.15, 22.32, and line (129). We have verified the result under the assumption that A, B, C is equitable. To remove the assumption, apply the result so far to the LR triple (127) in Lemma 16.31, made equitable by choosing the parameters (126) to satisfy (131). \square

Proposition 22.40. *Assume that A, B, C is bipartite and nontrivial. Let R denote a rotator for A, B, C and write $R = r \Omega_{\text{out}} + s \Omega_{\text{in}}$ with $r, s \in \mathbb{F}$. Then*

$$\begin{aligned} s \varphi'_d A_{\text{out}} R &= r \varphi''_1 R B_{\text{out}}, & s \varphi''_d B_{\text{out}} R &= r \varphi_1 R C_{\text{out}}, & s \varphi_d C_{\text{out}} R &= r \varphi'_1 R A_{\text{out}}, \\ r \varphi'_1 A_{\text{in}} R &= s \varphi''_d R B_{\text{in}}, & r \varphi''_1 B_{\text{in}} R &= s \varphi_d R C_{\text{in}}, & r \varphi_1 C_{\text{in}} R &= s \varphi'_d R A_{\text{in}}. \end{aligned}$$

Moreover for A, B, C equitable,

$$\begin{aligned} s A_{\text{out}} R &= r \rho_0 R B_{\text{out}}, & s B_{\text{out}} R &= r \rho'_0 R C_{\text{out}}, & s C_{\text{out}} R &= r \rho''_0 R A_{\text{out}}, \\ r \rho_0 A_{\text{in}} R &= s R B_{\text{in}}, & r \rho'_0 B_{\text{in}} R &= s R C_{\text{in}}, & r \rho''_0 C_{\text{in}} R &= s R A_{\text{in}}. \end{aligned}$$

Proof. To verify these equations, eliminate R using $R = r \Omega_{\text{out}} + s \Omega_{\text{in}}$ and evaluate the result using Definition 16.29 together with Proposition 22.39. \square

Lemma 22.41. *Assume that A, B, C is bipartite and nontrivial. Then Ω_{out} sends*

$$\eta \rightarrow \frac{(\eta, \tilde{\eta}')}{(\eta'', \tilde{\eta})} \eta'', \quad \eta' \rightarrow \frac{(\eta', \tilde{\eta}'')}{(\eta, \tilde{\eta}')} \eta, \quad \eta'' \rightarrow \frac{(\eta'', \tilde{\eta})}{(\eta', \tilde{\eta}'')} \eta'.$$

Proof. Similar to the proof of Lemma 22.23. \square

Recall the scalar θ from Definition 13.71.

Proposition 22.42. *Assume that A, B, C is bipartite and nontrivial. Then the following (i), (ii) hold.*

- (i) $\Omega_{\text{out}}^3 = \theta I$ on V_{out} .
- (ii) $\Omega_{\text{in}}^3 = \rho^{-1}\theta I$ on V_{in} , where $\rho = \varphi_1\varphi'_1\varphi''_1/(\varphi_d\varphi'_d\varphi''_d)$.

Proof. (i) Similar to the proof of Proposition 22.24.

(ii) By Proposition 22.39 we obtain $A\Omega_{\text{out}}^3 = \rho\Omega_{\text{in}}^3A$. For this equation apply each side to V_{out} and use the fact that $AV_{\text{out}} = V_{\text{in}}$. The result follows in view of (i) above. \square

Lemma 22.43. *Assume that A, B, C is bipartite and nontrivial. Let R denote a rotator for A, B, C and write $R = r\Omega_{\text{out}} + s\Omega_{\text{in}}$ with $r, s \in \mathbb{F}$. Then R is invertible if and only if r, s are nonzero.*

Proof. By Lemma 22.32. \square

Proposition 22.44. *Assume that A, B, C is bipartite and nontrivial. Let R denote an invertible rotator for A, B, C . Then*

$$\overline{AR} = R\overline{B}, \quad \overline{BR} = R\overline{C}, \quad \overline{CR} = R\overline{A}.$$

Proof. Use Theorem 20.3(iii) and the comment above that theorem, along with Proposition 22.40 and Lemma 22.43. \square

23 The reflectors for an LR triple

Throughout this section the following notation is in effect. Let V denote a vector space over \mathbb{F} with dimension $d + 1$. Let A, B, C denote an LR triple on V , with parameter array (44), idempotent data (48), trace data (50), and Toeplitz data (54). Recall the reflector antiautomorphism concept discussed in Proposition 6.1 and Definition 6.2. There are three reflectors associated with A, B, C ; the (A, B) -reflector, the (B, C) -reflector, and the (C, A) -reflector. We now consider how these reflectors behave. In order to keep things simple, throughout this section we assume that A, B, C is equitable.

Proposition 23.1. *Assume that A, B, C is equitable and nonbipartite, with standard rotator Ω . Then the following (i)–(iii) hold.*

- (i) *The (A, B) -reflector swaps A, B and fixes C . It swaps \mathbb{A}, \mathbb{B} and fixes \mathbb{C} . It fixes Ω . For $0 \leq i \leq d$ it fixes E_i and swaps E'_i, E''_i .*
- (ii) *The (B, C) -reflector swaps B, C and fixes A . It swaps \mathbb{B}, \mathbb{C} and fixes \mathbb{A} . It fixes Ω . For $0 \leq i \leq d$ it fixes E'_i and swaps E''_i, E_i .*
- (iii) *The (C, A) -reflector swaps C, A and fixes B . It swaps \mathbb{C}, \mathbb{A} and fixes \mathbb{B} . It fixes Ω . For $0 \leq i \leq d$ it fixes E''_i and swaps E_i, E'_i .*

Proof. (i) Denote the (A, B) -reflector by \dagger . The map \dagger swaps A, B by Proposition 6.1 and Definition 6.2. We have $\mathbb{A} = \sum_{i=0}^d \alpha_i A^i$ and $\mathbb{B} = \sum_{i=0}^d \alpha_i B^i$ by Proposition 21.13, so \dagger swaps \mathbb{A}, \mathbb{B} . To see that \dagger fixes Ω , use the first formula for Ω given in Theorem 22.18(i), along with Definition 7.1 and Lemma 7.10. From Theorem 22.18(v) we obtain $C = \Omega A \Omega^{-1}$ and $C = \Omega^{-1} B \Omega$. By these and since \dagger fixes Ω , we see that \dagger fixes C . Consequently \dagger fixes $\mathbb{C} = \sum_{i=0}^d \alpha_i C^i$. For $0 \leq i \leq d$ the map \dagger fixes E_i by Lemma 6.4. Also by Lemma 13.25 and Lemma 17.6(i) the map \dagger swaps E'_i, E''_i .

(ii), (iii) Use (i) above and Lemma 22.26(i). \square

Proposition 23.2. *Assume that A, B, C is equitable, bipartite, and nontrivial. Then the following (i)–(iii) hold.*

(i) *The (A, B) -reflector sends*

$$\begin{array}{lll} A_{\text{out}} \rightarrow B_{\text{in}}, & B_{\text{out}} \rightarrow A_{\text{in}}, & C_{\text{out}} \rightarrow (\rho_0''/\rho_0') C_{\text{in}}, \\ A_{\text{in}} \rightarrow B_{\text{out}}, & B_{\text{in}} \rightarrow A_{\text{out}}, & C_{\text{in}} \rightarrow (\rho_0'/\rho_0'') C_{\text{out}}. \end{array}$$

It swaps \mathbb{A}, \mathbb{B} and fixes \mathbb{C} . It fixes J and everything in \mathcal{R} . For $0 \leq i \leq d$ it fixes E_i and swaps E'_i, E''_i .

(ii) *The (B, C) -reflector sends*

$$\begin{array}{lll} B_{\text{out}} \rightarrow C_{\text{in}}, & C_{\text{out}} \rightarrow B_{\text{in}}, & A_{\text{out}} \rightarrow (\rho_0/\rho_0'') A_{\text{in}}, \\ B_{\text{in}} \rightarrow C_{\text{out}}, & C_{\text{in}} \rightarrow B_{\text{out}}, & A_{\text{in}} \rightarrow (\rho_0''/\rho_0) A_{\text{out}}. \end{array}$$

It swaps \mathbb{B}, \mathbb{C} and fixes \mathbb{A} . It fixes J and everything in \mathcal{R} . For $0 \leq i \leq d$ it fixes E'_i and swaps E''_i, E_i .

(iii) *The (C, A) -reflector sends*

$$\begin{array}{lll} C_{\text{out}} \rightarrow A_{\text{in}}, & A_{\text{out}} \rightarrow C_{\text{in}}, & B_{\text{out}} \rightarrow (\rho_0'/\rho_0) B_{\text{in}}, \\ C_{\text{in}} \rightarrow A_{\text{out}}, & A_{\text{in}} \rightarrow C_{\text{out}}, & B_{\text{in}} \rightarrow (\rho_0/\rho_0') B_{\text{out}}. \end{array}$$

It swaps \mathbb{C}, \mathbb{A} and fixes \mathbb{B} . It fixes J and everything in \mathcal{R} . For $0 \leq i \leq d$ it fixes E''_i and swaps E_i, E'_i .

Proof. (i) Denote the (A, B) -reflector by \dagger . The map \dagger swaps A, B by Proposition 6.1 and Definition 6.2. For $0 \leq i \leq d$ the map \dagger fixes E_i by Lemma 6.4. By this and Lemma 9.3(i), the map \dagger fixes J . By this and Lemma 9.9(i),(ii) the map \dagger sends $A_{\text{out}} \leftrightarrow B_{\text{in}}$ and $A_{\text{in}} \leftrightarrow B_{\text{out}}$. We have $\mathbb{A} = \sum_{i=0}^d \alpha_i A^i$ and $\mathbb{B} = \sum_{i=0}^d \alpha_i B^i$ by Proposition 21.13, so \dagger swaps \mathbb{A}, \mathbb{B} . We show that \dagger fixes everything in \mathcal{R} . By Proposition 22.33 it suffices to show that \dagger fixes Ω_{out} and Ω_{in} . To see that \dagger fixes Ω_{out} (resp. Ω_{in}), use the first formula for Ω_{out} (resp. Ω_{in}) given in Theorem 22.35(i) (resp. Theorem 22.36(i)), along with Definition 8.11 and Lemma 8.15. For $0 \leq i \leq d$ we show that \dagger swaps E'_i, E''_i . Pick an invertible $R \in \mathcal{R}$. By Definition 22.1, $E_i R = R E'_i$ and $E''_i R = R E_i$. In either equation, apply \dagger to each side and compare the results with the other equation. This shows that \dagger swaps E'_i, E''_i . Using this and $\mathbb{C} = \sum_{i=0}^d E''_{d-i} E'_i$ we find that \dagger fixes \mathbb{C} . To obtain the action of \dagger on $C_{\text{out}}, C_{\text{in}}$, we invoke

Proposition 22.40. Referring to that proposition, assume that r, s are nonzero, so that R is invertible, and consider the equations $sC_{\text{out}}R = r\rho_0''RA_{\text{out}}$ and $r\rho_0'B_{\text{in}}R = sRC_{\text{in}}$. In either equation, apply \dagger to each side and compare the results with the other equation. This shows that \dagger sends $C_{\text{out}} \mapsto (\rho_0''/\rho_0')C_{\text{in}}$ and $C_{\text{in}} \mapsto (\rho_0'/\rho_0'')C_{\text{out}}$.

(ii), (iii) Similar to the proof of (i) above. \square

Corollary 23.3. *Assume that A, B, C is equitable, bipartite, and nontrivial. Then the following (i)–(iii) hold:*

- (i) *the (A, B) -reflector swaps $\overline{A}, \overline{B}$ and fixes \overline{C} ;*
- (ii) *the (B, C) -reflector swaps $\overline{B}, \overline{C}$ and fixes \overline{A} ;*
- (iii) *the (C, A) -reflector swaps $\overline{C}, \overline{A}$ and fixes \overline{B} .*

Proof. Use Theorem 20.3(iii) and the comment above that theorem, along with Proposition 23.2. \square

24 Normalized LR triples with diameter at most 2

Our next general goal is to classify up to isomorphism the normalized LR triples. As a warmup, we consider the normalized LR triples with diameter at most 2. For the results in this section the proofs are routine, and left as an exercise.

Lemma 24.1. *Up to isomorphism, there exists a unique normalized LR triple over \mathbb{F} that has diameter 0. This LR triple is trivial.*

Lemma 24.2. *Up to isomorphism, there exists a unique normalized LR triple A, B, C over \mathbb{F} that has diameter 1. This LR triple is nonbipartite and $\varphi_1 = -1$. Moreover $a_0 = 1$ and $a_1 = -1$. With respect to an (A, B) -basis the matrices representing A, B, C and the standard rotator Ω are*

$$A: \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B: \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad C: \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad \Omega: \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Lemma 24.3. *We give a bijection from the set $\mathbb{F} \setminus \{0, -1\}$ to the set of isomorphism classes of normalized nonbipartite LR triples over \mathbb{F} that have diameter 2. For $q \in \mathbb{F} \setminus \{0, -1\}$ the corresponding LR triple A, B, C has parameters*

$$\begin{aligned} \varphi_1 &= -1 - q^{-1}, & \varphi_2 &= -1 - q, \\ \alpha_2 &= \frac{1}{1+q}, & \beta_2 &= \frac{q}{1+q}, \\ a_0 &= 1+q, & a_1 &= q^{-1} - q, & a_2 &= -1 - q^{-1}. \end{aligned}$$

With respect to an (A, B) -basis the matrices representing A, B, C and the standard rotator Ω are

$$\begin{aligned} A &: \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & B &: \begin{pmatrix} 0 & 0 & 0 \\ -1 - q^{-1} & 0 & 0 \\ 0 & -1 - q & 0 \end{pmatrix}, \\ C &: \begin{pmatrix} 1 + q & q & 0 \\ -1 - q & q^{-1} - q & q^{-1} \\ 0 & -1 - q^{-1} & -1 - q^{-1} \end{pmatrix}, & \Omega &: \begin{pmatrix} 1 & 1 & (1 + q)^{-1} \\ -1 - q^{-1} & -1 & 0 \\ 1 + q^{-1} & 0 & 0 \end{pmatrix}. \end{aligned}$$

Lemma 24.4. *We give a bijection from the set the 3-tuples*

$$(\rho_0, \rho'_0, \rho''_0) \in \mathbb{F}^3, \quad \rho_0 \rho'_0 \rho''_0 = -1 \quad (145)$$

to the set of isomorphism classes of normalized bipartite LR triples over \mathbb{F} that have diameter 2. For a 3-tuple $(\rho_0, \rho'_0, \rho''_0)$ in the set (145), the corresponding LR triple A, B, C has parameters

$$\begin{aligned} \varphi_1 &= -1/\rho_0, & \varphi'_1 &= -1/\rho'_0, & \varphi''_1 &= -1/\rho''_0, \\ \varphi_2 &= \rho_0, & \varphi'_2 &= \rho'_0, & \varphi''_2 &= \rho''_0. \end{aligned}$$

With respect to an (A, B) -basis the matrices representing A, B, C , the projector J , and the standard outer/inner rotators $\Omega_{\text{out}}, \Omega_{\text{in}}$ are

$$\begin{aligned} A &: \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & B &: \begin{pmatrix} 0 & 0 & 0 \\ -1/\rho_0 & 0 & 0 \\ 0 & \rho_0 & 0 \end{pmatrix}, & C &: \begin{pmatrix} 0 & 1/\rho''_0 & 0 \\ \rho''_0 & 0 & \rho''_0 \\ 0 & -1/\rho''_0 & 0 \end{pmatrix}, \\ J &: \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \Omega_{\text{out}} &: \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, & \Omega_{\text{in}} &: \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

25 The sequence $\{\rho_i\}_{i=0}^{d-1}$ is constrained

Throughout this section the following notation is in effect. Let V denote a vector space over \mathbb{F} with dimension $d + 1$. Let A, B, C denote a nontrivial LR triple on V , with parameter array (44), idempotent data (48), trace data (50), and Toeplitz data (54). We assume that A, B, C is equitable, so that $\alpha_i = \alpha'_i = \alpha''_i$ and $\beta_i = \beta'_i = \beta''_i$ for $0 \leq i \leq d$. For A, B, C nonbipartite we have the sequence $\{\rho_i\}_{i=0}^{d-1}$ from Definition 17.7, and for A, B, C bipartite we have the sequences $\{\rho_i\}_{i=0}^{d-1}$, $\{\rho'_i\}_{i=0}^{d-1}$, $\{\rho''_i\}_{i=0}^{d-1}$ from Definition 17.15. Our next goal is to show that these sequences are constrained, in the sense of Definition 11.1.

Lemma 25.1. *Assume that A, B, C is equitable. Then the following (i)–(iii) hold.*

(i) *For $d \geq 2$,*

$$\rho_i = \alpha_0 \beta_2 \varphi_i + \alpha_1 \beta_1 \varphi_{i+1} + \alpha_2 \beta_0 \varphi_{i+2} \quad (0 \leq i \leq d - 1), \quad (146)$$

$$0 = \alpha_0 \beta_2 + \alpha_1 \beta_1 + \alpha_2 \beta_0. \quad (147)$$

(ii) For $d \geq 3$,

$$0 = \alpha_0\beta_3\varphi_{i-2} + \alpha_1\beta_2\varphi_{i-1} + \alpha_2\beta_1\varphi_i + \alpha_3\beta_0\varphi_{i+1} \quad (2 \leq i \leq d), \quad (148)$$

$$0 = \alpha_0\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_0. \quad (149)$$

(iii) For A, B, C bipartite and $d \geq 4$,

$$0 = \alpha_0\beta_4\varphi_{i-2} + \alpha_2\beta_2\varphi_i + \alpha_4\beta_0\varphi_{i+2} \quad (2 \leq i \leq d-1), \quad (150)$$

$$0 = \alpha_0\beta_4 + \alpha_2\beta_2 + \alpha_4\beta_0. \quad (151)$$

Proof. (i) Line (146) is from the first displayed equation in Proposition 14.6, along with Lemma 17.6(i). Line (147) is from (42).

(ii) Line (148) is from the first displayed equation in Proposition 14.7 (with $r = 3$). Line (149) is from (42).

(iii) Similar to (ii) above, but also use Lemma 16.6. \square

As we proceed, we will consider the bipartite and nonbipartite cases separately. We begin with the nonbipartite case.

Lemma 25.2. *Assume that A, B, C is nonbipartite, equitable, and $d \geq 2$. Then for $0 \leq i \leq d-1$,*

$$\rho_i = \alpha_0\beta_2(\varphi_i - \varphi_{i+1}) - \alpha_2\beta_0(\varphi_{i+1} - \varphi_{i+2}). \quad (152)$$

Proof. Subtract φ_{i+1} times (147) from (146). \square

Definition 25.3. *Assume that A, B, C is nonbipartite, equitable, and $d \geq 3$. Define*

$$a = \alpha_0\beta_3, \quad b = \alpha_0\beta_3 + \alpha_1\beta_2 = -\alpha_2\beta_1 - \alpha_3\beta_0, \quad c = -\alpha_3\beta_0. \quad (153)$$

Lemma 25.4. *Assume that A, B, C is nonbipartite, equitable, and $d \geq 3$. Then for $2 \leq i \leq d$,*

$$0 = a(\varphi_{i-2} - \varphi_{i-1}) + b(\varphi_{i-1} - \varphi_i) + c(\varphi_i - \varphi_{i+1}), \quad (154)$$

where a, b, c are from (153).

Proof. To verify (154), eliminate a, b, c using (153), and compare the result with (148). \square

Lemma 25.5. *Assume that A, B, C is nonbipartite, equitable, and $d \geq 3$. Then for $1 \leq i \leq d-2$,*

$$0 = a\rho_{i-1} + b\rho_i + c\rho_{i+1}, \quad (155)$$

where a, b, c are from (153).

Proof. To verify (155), eliminate $\rho_{i-1}, \rho_i, \rho_{i+1}$ using Lemma 25.2, and evaluate the result using Lemma 25.4. \square

Lemma 25.6. *Assume that A, B, C is nonbipartite, equitable, and $d \geq 3$. Then the scalars a, b, c from Definition 25.3 are not all zero.*

Proof. Recall from Lemmas 13.46, 16.5 that $\alpha_0 = 1 = \beta_0$ and $\alpha_1 = -\beta_1$ is nonzero. Suppose that each of a, b, c is zero. Using (153) we obtain $\alpha_3 = 0, \beta_3 = 0, \alpha_2 = 0, \beta_2 = 0$. Now in (152) the right-hand side is zero and the left-hand side is nonzero, for a contradiction. The result follows. \square

Proposition 25.7. *Assume that A, B, C is nonbipartite and equitable. Then the sequence $\{\rho_i\}_{i=0}^{d-1}$ is constrained.*

Proof. We verify that $\{\rho_i\}_{i=0}^{d-1}$ satisfies the conditions (i), (ii) of Definition 11.1. Definition 11.1(i) holds by Lemma 17.8. If $d \leq 2$ then Definition 11.1(ii) holds vacuously, and if $d \geq 3$ then Definition 11.1(ii) holds by Lemmas 25.5, 25.6. \square

Lemma 25.8. *Assume that A, B, C is nonbipartite and equitable, but the sequence $\{\rho_i\}_{i=0}^{d-1}$ is not geometric. Then d is even and at least 4.*

Proof. By Lemma 11.6 and Proposition 25.7. \square

We turn our attention to bipartite LR triples.

Lemma 25.9. *Assume that A, B, C is bipartite, equitable, and $d \geq 2$. Then for $0 \leq i \leq d-1$,*

$$\rho_i = \alpha_0 \beta_2 (\varphi_i - \varphi_{i+2}), \quad \rho'_i = \alpha_0 \beta_2 (\varphi'_i - \varphi'_{i+2}), \quad \rho''_i = \alpha_0 \beta_2 (\varphi''_i - \varphi''_{i+2}). \quad (156)$$

Proof. To verify the equation on the left in (156), set $\alpha_1 = 0, \beta_1 = 0$ in Lemma 25.1(i). The other two equations in (156) are similarly verified. \square

Lemma 25.10. *Assume that A, B, C is bipartite, equitable, and $d \geq 4$. Then for $2 \leq i \leq d-1$,*

$$\alpha_0 \beta_4 (\varphi_{i-2} - \varphi_i) = \alpha_4 \beta_0 (\varphi_i - \varphi_{i+2}), \quad (157)$$

$$\alpha_0 \beta_4 (\varphi'_{i-2} - \varphi'_i) = \alpha_4 \beta_0 (\varphi'_i - \varphi'_{i+2}), \quad (158)$$

$$\alpha_0 \beta_4 (\varphi''_{i-2} - \varphi''_i) = \alpha_4 \beta_0 (\varphi''_i - \varphi''_{i+2}). \quad (159)$$

Proof. To obtain (157), subtract φ_i times (151) from (150). Equations (158), (159) are similarly obtained. \square

Lemma 25.11. *Assume that A, B, C is bipartite, equitable, and $d \geq 4$. Then for $1 \leq i \leq d-2$,*

$$\alpha_0 \beta_4 \rho_{i-1} = \alpha_4 \beta_0 \rho_{i+1}, \quad \alpha_0 \beta_4 \rho'_{i-1} = \alpha_4 \beta_0 \rho'_{i+1}, \quad \alpha_0 \beta_4 \rho''_{i-1} = \alpha_4 \beta_0 \rho''_{i+1}. \quad (160)$$

Proof. Use Lemmas 25.9, 25.10. \square

Proposition 25.12. *Assume that A, B, C is bipartite, equitable, and nontrivial. Then the sequences $\{\rho_i\}_{i=0}^{d-1}, \{\rho'_i\}_{i=0}^{d-1}, \{\rho''_i\}_{i=0}^{d-1}$ are constrained.*

Proof. We verify that $\{\rho_i\}_{i=0}^{d-1}, \{\rho'_i\}_{i=0}^{d-1}, \{\rho''_i\}_{i=0}^{d-1}$ satisfy the conditions (i), (ii) of Definition 11.1. Definition 11.1(i) holds by Lemma 17.16. Recall that d is even. If $d = 2$ then Definition 11.1(ii) holds vacuously, and if $d \geq 4$ then Definition 11.1(ii) holds by Lemma 25.11 and since $\alpha_4 \neq 0, \beta_4 \neq 0$ by Lemma 16.6. \square

26 The classification of normalized LR triples; an overview

Throughout this section assume $d \geq 2$. Our next goal is to classify up to isomorphism the normalized LR triples over \mathbb{F} that have diameter d . We now describe our strategy. Consider a normalized LR triple A, B, C over \mathbb{F} that has parameter array (44). Recall the sequence $\{\rho_i\}_{i=0}^{d-1}$ from Definition 17.7. We place A, B, C into one of four families as follows:

family name	family definition	d restriction
$\text{NBWeyl}_d(\mathbb{F})$	over \mathbb{F} ; diameter d ; nonbipartite; normalized; there exist scalars a, b, c in \mathbb{F} that are not all zero such that $a + b + c = 0$ and $a\varphi_{i-1} + b\varphi_i + c\varphi_{i+1} = 0$ for $1 \leq i \leq d$	
$\text{NBG}_d(\mathbb{F})$	over \mathbb{F} ; diameter d ; nonbipartite; normalized; not in $\text{NBWeyl}_d(\mathbb{F})$; the sequence $\{\rho_i\}_{i=0}^{d-1}$ is geometric	
$\text{NBNG}_d(\mathbb{F})$	over \mathbb{F} ; diameter d ; nonbipartite; normalized; the sequence $\{\rho_i\}_{i=0}^{d-1}$ is not geometric	d even; $d \geq 4$
$\text{B}_d(\mathbb{F})$	over \mathbb{F} ; diameter d ; bipartite; normalized	d even

As we will show in Lemma 27.6, if A, B, C is contained in $\text{NBWeyl}_d(\mathbb{F})$ then $\{\rho_i\}_{i=0}^{d-1}$ is geometric. By this and Lemmas 16.6, 25.8 the LR triple A, B, C falls into exactly one of the four families.

Over the next four sections, we classify up to isomorphism the LR triples in each family.

27 The classification of LR triples in $\text{NBWeyl}_d(\mathbb{F})$

In this section we classify up to isomorphism the LR triples in $\text{NBWeyl}_d(\mathbb{F})$, for $d \geq 2$. We first describe some examples.

Example 27.1. The LR triple $\text{NBWeyl}_d^+(\mathbb{F}; j, q)$ is over \mathbb{F} , diameter d , nonbipartite, normalized, and satisfies

$$\begin{aligned}
 & d \geq 2; \quad d \text{ is even}; \quad j \in \mathbb{Z}, \quad 0 \leq j < d/2; \quad 0 \neq q \in \mathbb{F}; \\
 & \text{if } \text{Char}(\mathbb{F}) \neq 2 \text{ then } q \text{ is a primitive } (2d+2)\text{-root of unity;} \\
 & \text{if } \text{Char}(\mathbb{F}) = 2 \text{ then } q \text{ is a primitive } (d+1)\text{-root of unity;} \\
 & \varphi_i = \frac{(1 + q^{2j+1})^2(1 - q^{-2i})}{q^{2j+1}(q - q^{-1})^2} \quad (1 \leq i \leq d).
 \end{aligned}$$

Example 27.2. The LR triple $\text{NBWeyl}_d^-(\mathbb{F}; j, q)$ is over \mathbb{F} , diameter d , nonbipartite, nor-

malized, and satisfies

$$\begin{aligned} \text{Char}(\mathbb{F}) &\neq 2; & d &\geq 3; & d &\text{ is odd;} \\ j &\in \mathbb{Z}, \quad 0 \leq j < (d-1)/4; & 0 &\neq q \in \mathbb{F}; \\ q &\text{ is a primitive } (2d+2)\text{-root of unity;} \\ \varphi_i &= \frac{(1+q^{2j+1})^2(1-q^{-2i})}{q^{2j+1}(q-q^{-1})^2} & (1 \leq i \leq d). \end{aligned}$$

Example 27.3. The LR triple $\text{NBWeyl}_d^-(\mathbb{F}; t)$ is over \mathbb{F} , diameter d , nonbipartite, normalized, and satisfies

$$\begin{aligned} \text{Char}(\mathbb{F}) &\neq 2; & d &\geq 5; & d &\equiv 1 \pmod{4}; \\ 0 &\neq t \in \mathbb{F}; & t &\text{ is a primitive } (d+1)\text{-root of unity;} \\ \varphi_i &= \frac{2t(1-t^i)}{(1-t)^2} & (1 \leq i \leq d). \end{aligned}$$

Lemma 27.4. *For the LR triples in Examples 27.1–27.3, (i) they exist; (ii) they are contained in $\text{NBWeyl}_d(\mathbb{F})$; (iii) they are mutually nonisomorphic.*

Proof. (i) In Examples 27.1, 27.2 we see an integer j . For Example 27.3 define an integer $j = (d-1)/4$. In Examples 27.1, 27.2 we see a parameter $q \in \mathbb{F}$. For Example 27.3, define $q \in \overline{\mathbb{F}}$ such that $t = q^{-2}$. In each of Examples 27.1–27.3 the pair d, q is standard. For each example we use the data d, j, q and Proposition 15.31 to get an LR triple over $\overline{\mathbb{F}}$ that has q -Weyl type. This LR triple is nonbipartite, since its first Toeplitz number is nonzero. Normalize this LR triple and apply Lemma 15.10 to get the desired LR triple over \mathbb{F} .

(ii) Let A, B, C denote an LR triple listed in Examples 27.1–27.3. By assumption A, B, C is over \mathbb{F} , diameter d , nonbipartite, and normalized. Define $a = 1, b = -1 - q^2, c = q^2$, where $t = q^{-2}$ in Example 27.3. Then $a + b + c = 0$, and $a\varphi_{i-1} + b\varphi_i + c\varphi_{i+1} = 0$ for $1 \leq i \leq d$. Therefore A, B, C is contained in $\text{NBWeyl}_d(\mathbb{F})$.

(iii) Among the LR triples listed in Examples 27.1–27.3, no two have the same parameter array. Therefore no two are isomorphic. \square

Theorem 27.5. *For $d \geq 2$, each LR triple in $\text{NBWeyl}_d(\mathbb{F})$ is isomorphic to a unique LR triple listed in Examples 27.1–27.3.*

Proof. Let A, B, C denote an LR triple in $\text{NBWeyl}_d(\mathbb{F})$, with parameter array (44) and Toeplitz data (54). By assumption there exist scalars a, b, c in \mathbb{F} that are not all zero, such that $a + b + c = 0$ and $a\varphi_{i-1} + b\varphi_i + c\varphi_{i+1} = 0$ for $1 \leq i \leq d$. Setting $i = 1$ and $\varphi_0 = 0$ we obtain $b\varphi_1 + c\varphi_2 = 0$. Setting $i = d$ and $\varphi_{d+1} = 0$ we obtain $a\varphi_{d-1} + b\varphi_d = 0$. By these comments each of a, b, c is nonzero. Define a polynomial $g \in \mathbb{F}[\lambda]$ by $g(\lambda) = a + b\lambda + c\lambda^2$. Observe that $g(1) = 0$, so there exists $t \in \mathbb{F}$ such that $g(\lambda) = c(\lambda-1)(\lambda-t)$. We have $ct = a$, so $t \neq 0$. Assume for the moment that $t = 1$. By construction there exists $u, v \in \mathbb{F}$ such that $\varphi_i = u + vi$ for $0 \leq i \leq d+1$. setting $i = 0$ and $\varphi_0 = 0$ we obtain $u = 0$. Consequently $\varphi_i = vi$ for $0 \leq i \leq d+1$, and $v \neq 0$. Fix a square root $v^{1/2} \in \overline{\mathbb{F}}$. Define an LR triple A^\vee, B^\vee, C^\vee over $\overline{\mathbb{F}}$ by

$$A^\vee = Av^{-1/2}, \quad B^\vee = Bv^{-1/2}, \quad C^\vee = Cv^{-1/2}.$$

By Lemma 13.7 this LR triple has parameter array

$$\varphi_i^\vee = (\varphi_i')^\vee = (\varphi_i'')^\vee = \varphi_i/v = i \quad (1 \leq i \leq d).$$

Now by Definition 15.17 the LR triple A^\vee, B^\vee, C^\vee has Weyl type. Consider its first Toeplitz number α_1^\vee . On one hand, by Lemma 13.48 and the construction, $\alpha_1^\vee = \alpha_1 v^{1/2} = v^{1/2}$. On the other hand, by Lemma 15.21, $\alpha_1^\vee = 0$. This is a contradiction, so $t \neq 1$. The polynomial $g(\lambda) = c(\lambda-1)(\lambda-t)$ has distinct roots. Therefore there exist $u, v \in \mathbb{F}$ such that $\varphi_i = u + vt^i$ for $0 \leq i \leq d+1$. Setting $i = 0$ and $\varphi_0 = 0$ we obtain $0 = u + v$. Consequently $\varphi_i = u(1 - t^i)$ for $0 \leq i \leq d+1$, and $u \neq 0$. Fix square roots $u^{1/2}, t^{1/2} \in \overline{\mathbb{F}}$. Define $q = t^{-1/2}$. By construction $q \neq 0$, $t = q^{-2}$, and $q^2 \neq 1$. Define an LR triple A^\vee, B^\vee, C^\vee over $\overline{\mathbb{F}}$ by

$$A^\vee = Au^{-1/2}, \quad B^\vee = Bu^{-1/2}, \quad C^\vee = Cu^{-1/2}.$$

By Lemma 13.7 this LR triple has parameter array

$$\varphi_i^\vee = (\varphi_i')^\vee = (\varphi_i'')^\vee = \varphi_i/u = 1 - q^{-2i} \quad (1 \leq i \leq d).$$

Now by Definition 15.24, the LR triple A^\vee, B^\vee, C^\vee has q -Weyl type. Replacing q by $-q$ if necessary, we may assume by Lemma 4.16 that the pair d, q is standard in the sense of Definition 4.14. By that definition, if $\text{Char}(\mathbb{F}) \neq 2$ then q is a primitive $(2d+2)$ -root of unity. Moreover if $\text{Char}(\mathbb{F}) = 2$, then d is even and q is a primitive $(d+1)$ -root of unity. Consider the first Toeplitz number α_1^\vee . By Lemma 15.29 there exists an integer j ($0 \leq j \leq d$) such that

$$\alpha_1^\vee = \frac{q^{j+1/2} + q^{-j-1/2}}{q - q^{-1}}.$$

By Lemma 13.48 and the construction, $\alpha_1^\vee = u^{1/2}$. Therefore $u = (\alpha_1^\vee)^2$. By these comments

$$u = \frac{(1 + q^{2j+1})^2}{q^{2j+1}(q - q^{-1})^2}. \quad (161)$$

Replacing j by $d-j$ corresponds to replacing $u^{1/2}$ by $-u^{1/2}$, and this move leaves u invariant. Replacing j by $d-j$ if necessary, we may assume without loss that $j \leq d/2$. Note that $j \neq d/2$; otherwise $1 + q^{2j+1} = 0$ which contradicts (161). Therefore $j < d/2$. Assume for the moment that $d = 1 + 4j$. Then $q^{2j+1} + q^{-2j-1} = 0$. In this case (161) reduces to $u = 2/(q - q^{-1})^2$, or in other words $u = 2t/(1-t)^2$. Now d, t satisfy the requirements of Example 27.3, so A, B, C is isomorphic to $\text{NBWeyl}_d^-(\mathbb{F}; t)$. For the rest of this proof, assume that $d \neq 1 + 4j$. We show $q \in \mathbb{F}$. Define $f = q^{2j+1} + q^{-2j-1}$ and note by (161) that $f + 2 = uq^2(1 - q^{-2})^2$. By this and $u, q^2 \in \mathbb{F}$ we find $f \in \mathbb{F}$. Also, using $q^2 \in \mathbb{F}$ we obtain $fq = q^{2j+2} + q^{-2j} \in \mathbb{F}$. We have $f \neq 0$ since $d \neq 1 + 4j$. By these comments $q = fq/f \in \mathbb{F}$. For the moment assume that d is even. Then d, j, q meet the requirements of Example 27.1, so A, B, C is isomorphic to $\text{NBWeyl}_d^+(\mathbb{F}; j, q)$. Next assume that d is odd. We mentioned earlier that the pair d, q is standard. The pair d, q remains standard if we replace q by $-q$. Consider what happens if we replace q by $-q$ and j by $(d-1)/2 - j$. By (161) this replacement has no effect on u . Making this adjustment if necessary, we may

assume without loss that $j < (d-1)/4$. Now d, j, q meet the requirements of Example 27.2, so A, B, C is isomorphic to $\text{NBWeyl}_d^-(\mathbb{F}; j, q)$. We have shown that A, B, C is isomorphic to at least one of the LR triples listed in Examples 27.1–27.3. The result follows from this and Lemma 27.4(iii). \square

Lemma 27.6. *Assume $d \geq 2$. Let A, B, C denote an LR triple in $\text{NBWeyl}_d(\mathbb{F})$. Then for $0 \leq i \leq d-1$ the scalar ρ_i from Definition 17.7 satisfies*

case	$\text{NBWeyl}_d^+(\mathbb{F}; j, q)$	$\text{NBWeyl}_d^-(\mathbb{F}; j, q)$	$\text{NBWeyl}_d^-(\mathbb{F}; t)$
ρ_i	$-q^{-2i-2}$	$-q^{-2i-2}$	$-t^{i+1}$

Moreover, the sequence $\{\rho_i\}_{i=0}^{d-1}$ is geometric.

Proof. Compute $\rho_i = \varphi_{i+1}/\varphi_{d-i}$ using the data in Examples 27.1–27.3. \square

28 The classification of LR triples in $\text{NBG}_d(\mathbb{F})$

In this section we classify up to isomorphism the LR triples in $\text{NBG}_d(\mathbb{F})$, for $d \geq 2$. We first describe some examples.

Example 28.1. The LR triple $\text{NBG}_d(\mathbb{F}; q)$ is over \mathbb{F} , diameter d , nonbipartite, normalized, and satisfies

$$\begin{aligned}
 d &\geq 2; & 0 &\neq q \in \mathbb{F}; \\
 q^i &\neq 1 \quad (1 \leq i \leq d); & q^{d+1} &\neq -1; \\
 \rho_i &= \frac{q(q^i - 1)(q^{i-d-1} - 1)}{(q - 1)^2} & (1 \leq i \leq d).
 \end{aligned}$$

Example 28.2. The LR triple $\text{NBG}_d(\mathbb{F}; 1)$ is over \mathbb{F} , diameter d , nonbipartite, normalized, and satisfies

$$\begin{aligned}
 d &\geq 2; & \text{Char}(\mathbb{F}) &\text{ is 0 or greater than } d; \\
 \rho_i &= i(i - d - 1) & (1 \leq i \leq d).
 \end{aligned}$$

Lemma 28.3. *For the LR triples in Examples 28.1, 28.2, (i) they exist; (ii) they are contained in $\text{NBG}_d(\mathbb{F})$; (iii) they are mutually nonisomorphic.*

Proof. (i) In Example 28.1 we see a parameter $q \in \mathbb{F}$. For Example 28.2 define $q = 1$. Using q we construct an LR triple A, B, C as follows. For notational convenience define $a_i = \varphi_{d-i+1} - \varphi_{d-i}$ for $0 \leq i \leq d$, where $\varphi_0 = 0$ and $\varphi_{d+1} = 0$. Let V denote a vector space over \mathbb{F} with dimension $d+1$. Let $\{v_i\}_{i=0}^d$ denote a basis for V . Define A, B, C in $\text{End}(V)$ such that the matrices representing A, B, C with respect to $\{v_i\}_{i=0}^d$ are given by the first row of the table in Proposition 13.39. Here $\varphi'_i = \varphi_i$ and $\varphi''_i = \varphi_i$ for $1 \leq i \leq d$. We show that A, B, C is an LR triple on V . We first show that A, B is an LR pair on V . Let $\{V_i\}_{i=0}^d$ denote the decomposition of V induced by the basis $\{v_i\}_{i=0}^d$. Using the matrices defining A, B we find that $\{V_i\}_{i=0}^d$ is lowered by A and raised by B . Therefore A, B is an LR pair

on V . Next we show that B, C is an LR pair on V . Define the scalars $\{\alpha_i\}_{i=0}^d$ by $\alpha_0 = 1$ and $\alpha_{i-1}/\alpha_i = \sum_{k=0}^{i-1} q^k$ for $1 \leq i \leq d$. Define $\mathbb{B} = \sum_{i=0}^d \alpha_i B^i$. Note that $\mathbb{B}B = B\mathbb{B}$. With respect to the basis $\{v_i\}_{i=0}^d$ the matrix representing \mathbb{B} is lower triangular, with each diagonal entry 1. Therefore \mathbb{B} is invertible. Observe that $\{\mathbb{B}V_{d-i}\}_{i=0}^d$ is a decomposition of V that is lowered by B . Using the matrices defining B, C one checks that $\{\mathbb{B}V_{d-i}\}_{i=0}^d$ is raised by C . By these comments B, C is an LR pair on V . Next we show that C, A is an LR pair on V . Define $\mathbb{A}^\S = \sum_{i=0}^d \beta_i A^i$, where $\beta_0 = 1$ and $\beta_{i-1}/\beta_i = -\sum_{k=0}^{i-1} q^{-k}$ for $1 \leq i \leq d$. Note that $\mathbb{A}^\S A = A\mathbb{A}^\S$. With respect to the basis $\{v_i\}_{i=0}^d$ the matrix representing \mathbb{A}^\S is upper triangular and Toeplitz, with parameters $\{\beta_i\}_{i=0}^d$. Therefore \mathbb{A}^\S is invertible. (In fact \mathbb{A}^\S is the inverse of $\mathbb{A} = \sum_{i=0}^d \alpha_i A^i$, although we do not need this result). Observe that $\{\mathbb{A}^\S V_{d-i}\}_{i=0}^d$ is a decomposition of V that is raised by A . Using the matrices defining A, C one checks that $\{\mathbb{A}^\S V_{d-i}\}_{i=0}^d$ is lowered by C . By these comments C, A is an LR pair on V . We have shown that A, B, C is an LR triple on V . Using the matrices defining A, B, C we find that this LR triple is the desired one.

(ii) Let A, B, C denote an LR triple listed in Examples 28.1, 28.2. By assumption A, B, C is over \mathbb{F} , diameter d , nonbipartite, and normalized. We check that A, B, C is not in $\text{NBWeyl}_d(\mathbb{F})$. The matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ \varphi_0 & \varphi_1 & \varphi_2 \\ \varphi_1 & \varphi_2 & \varphi_3 \end{pmatrix}$$

has determinant $-(q+1)(q^{d+1}+1)q^{2-2d}$ for $\text{NBG}_d(\mathbb{F}; q)$, and -4 for $\text{NBG}_d(\mathbb{F}; 1)$. In each case the determinant is nonzero. Consequently, there does not exist a, b, c in \mathbb{F} that are not all zero such that $a+b+c=0$ and $a\varphi_{i-1}+b\varphi_i+c\varphi_{i+1}=0$ for $1 \leq i \leq d$. Therefore A, B, C is not in $\text{NBWeyl}_d(\mathbb{F})$. We check that the sequence $\{\rho_i\}_{i=0}^{d-1}$ from Definition 17.7 is geometric. For $0 \leq i \leq d-1$ the scalar $\rho_i = \varphi_{i+1}/\varphi_{d-i}$ is equal to q^{2i-d+1} for $\text{NBG}_d(\mathbb{F}; q)$, and 1 for $\text{NBG}_d(\mathbb{F}; 1)$. Therefore $\{\rho_i\}_{i=0}^{d-1}$ is geometric. We have shown that A, B, C is contained in $\text{NBG}_d(\mathbb{F})$.

(iii) Among the LR triples listed in Examples 28.1, 28.2 no two have the same parameter array. Therefore no two are isomorphic. \square

Theorem 28.4. *For $d \geq 2$, each LR triple in $\text{NBG}_d(\mathbb{F})$ is isomorphic to a unique LR triple listed in Examples 28.1, 28.2.*

Proof. Let A, B, C denote an LR triple in $\text{NBG}_d(\mathbb{F})$, with parameter array (44) and Toeplitz data (54). Recall the sequence $\{\rho_i\}_{i=0}^{d-1}$ from Definition 17.7. This sequence is constrained by Proposition 25.7, and geometric by the definition of $\text{NBG}_d(\mathbb{F})$. Therefore there exists $0 \neq r \in \mathbb{F}$ such that

$$\rho_i = \rho_0 r^i \quad (0 \leq i \leq d-1), \quad (162)$$

$$\rho_0^2 = r^{1-d}. \quad (163)$$

By assumption A, B, C is nonbipartite and normalized, so $\alpha_1 = 1$ and $\beta_1 = -1$. Also $\alpha_2 + \beta_2 = 1$ from above Lemma 12.5. Define

$$q = \begin{cases} \beta_2/\alpha_2 & \text{if } \alpha_2 \neq 0; \\ \infty & \text{if } \alpha_2 = 0. \end{cases} \quad (164)$$

Conceivably $q = 0$ or $q = 1$. Using $\beta_2 = q\alpha_2$ and $\alpha_2 + \beta_2 = 1$, we obtain $q \neq -1$ and

$$\alpha_2 = \frac{1}{1+q}, \quad \beta_2 = \frac{q}{1+q}. \quad (165)$$

If $q = \infty$ then $\beta_2 = 1$. Evaluating (152) using (165) we obtain

$$\rho_i = \frac{q\varphi_i - (q+1)\varphi_{i+1} + \varphi_{i+2}}{q+1} \quad (0 \leq i \leq d-1). \quad (166)$$

If $q = \infty$ then (166) becomes $\rho_i = \varphi_i - \varphi_{i+1}$ for $0 \leq i \leq d-1$. Until further notice, assume that $q \neq 0$, $q \neq \infty$, and $1, q, r$ are mutually distinct. Define

$$L = \frac{(q+1)\rho_0}{(q-r)(1-r)}. \quad (167)$$

Note that $L \neq 0$. Since $q \neq 1$, there exist $H, K \in \mathbb{F}$ such that $\varphi_i = H + Kq^i + Lr^i$ for $i = 0$ and $i = 1$. Using (162), (166) and induction on i , we obtain

$$\varphi_i = H + Kq^i + Lr^i \quad (0 \leq i \leq d+1). \quad (168)$$

We have $K \neq 0$; otherwise $r\varphi_{i-1} - (r+1)\varphi_i + \varphi_{i+1} = 0$ ($1 \leq i \leq d$), putting A, B, C in $\text{NBWeyl}_d(\mathbb{F})$ for a contradiction. Since $\varphi_0 = 0$ and $\varphi_{d+1} = 0$,

$$0 = H + K + L, \quad 0 = H + Kq^{d+1} + Lr^{d+1}. \quad (169)$$

For $0 \leq i \leq d+1$ define

$$\Delta_i = \varphi_i - \rho_0 r^{i-1} \varphi_{d-i+1}. \quad (170)$$

We claim $\Delta_i = 0$. This is the case for $i = 0$ and $i = d+1$, since $\varphi_0 = 0$ and $\varphi_{d+1} = 0$. For $1 \leq i \leq d$ we have $\Delta_i = \varphi_i - \rho_{i-1} \varphi_{d-i+1}$ by (162), and this is zero by (128). The claim is proven. For $0 \leq i \leq d+1$, in the equation $\Delta_i = 0$ we evaluate the left-hand side using (168), (170) to find that the following linear combination is zero:

term	1	q^i	r^i	$(r/q)^i$
coefficient	$H - L\rho_0 r^d$	K	$L - H\rho_0 r^{-1}$	$-K\rho_0 q^{d+1} r^{-1}$

By assumption $1, q, r$ are mutually distinct. Also $K \neq 0$ and $d \geq 2$. We show $r = q^2$. Assume $r \neq q^2$. Then $1, q, r, r/q$ are mutually distinct. Setting $i = 0, 1, 2, 3$ in the above table, we obtain a 4×4 homogeneous linear system with coefficient matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & q & r & r/q \\ 1 & q^2 & r^2 & (r/q)^2 \\ 1 & q^3 & r^3 & (r/q)^3 \end{pmatrix}.$$

This matrix is Vandermonde and hence invertible. Therefore each coefficient in the table is zero. The coefficient K is nonzero, for a contradiction. Consequently $r = q^2$. From further examination of the coefficients in the table,

$$H = L\rho_0 r^d, \quad K = K\rho_0 q^{d+1} r^{-1}, \quad L = H\rho_0 r^{-1}. \quad (171)$$

By (171) and $r = q^2$ we find $\rho_0 = q^{1-d}$ and $H = Lq^{d+1}$. By these comments and (167) we obtain

$$H = q(q-1)^{-2}, \quad L = q^{-d}(q-1)^{-2}. \quad (172)$$

Evaluating (168) using (172) and $K = -H - L$, we obtain

$$\varphi_i = \frac{q(q^i - 1)(q^{i-d-1} - 1)}{(q-1)^2} \quad (1 \leq i \leq d). \quad (173)$$

From (173) and since the $\{\varphi_i\}_{i=1}^d$ are nonzero, we obtain $q^i \neq 1$ ($1 \leq i \leq d$). Note that $q^{d+1} \neq -1$; otherwise (173) becomes $\varphi_i = q(1 - q^{2i})(q-1)^{-2}$ ($1 \leq i \leq d$), forcing $q\varphi_{i-1} - (q + q^{-1})\varphi_i + q^{-1}\varphi_{i+1} = 0$ ($1 \leq i \leq d$), putting A, B, C in $\text{NBWeyl}_d(\mathbb{F})$ for a contradiction. We have met the requirements of Example 28.1, so A, B, C is isomorphic to $\text{NBG}_d(\mathbb{F}; q)$. We are done with the case in which $q \neq 0$, $q \neq \infty$, and $1, q, r$ are mutually distinct. Until further notice, assume that $1 = q = r$. We have $1 = q \neq -1$, so $\text{Char}(\mathbb{F}) \neq 2$. By (162), (163) we obtain $\rho_i = \rho_0$ for $0 \leq i \leq d-1$, and $\rho_0^2 = 1$. By (166),

$$2\rho_0 = \varphi_i - 2\varphi_{i+1} + \varphi_{i+2} \quad (0 \leq i \leq d-1). \quad (174)$$

Define $Q = \varphi_1 - \rho_0$, and note that $\varphi_i = i(Q + \rho_0 i)$ for $i = 0$ and $i = 1$. By (174) and induction on i ,

$$\varphi_i = i(Q + \rho_0 i) \quad (0 \leq i \leq d+1). \quad (175)$$

Mimicking the argument below (170), we find that for $0 \leq i \leq d+1$,

$$0 = \varphi_i - \rho_0 \varphi_{d-i+1}. \quad (176)$$

Evaluate the right-hand side of (176) using (175) and $\rho_0^2 = 1$, to find that the following linear combination is zero:

term	1	i	i^2
coefficient	$-(d+1)(d+1+\rho_0 Q)$	$2(d+1)+Q(1+\rho_0)$	$\rho_0 - 1$

Since $d \geq 2$, each coefficient in the table is zero. Therefore $\rho_0 = 1$ and $Q = -d-1$. By this and (175),

$$\varphi_i = i(i-d-1) \quad (1 \leq i \leq d). \quad (177)$$

By (177) and since $\{\varphi_i\}_{i=1}^d$ are nonzero, $\text{Char}(\mathbb{F})$ is 0 or greater than d . We have met the requirements of Example 28.2, so A, B, C is isomorphic to $\text{NBG}_d(\mathbb{F}; 1)$. We are done with the case $1 = q = r$. The remaining cases are (a) $q = 0$ and $r \neq 1$; (b) $q = 0$ and $r = 1$; (c) $q = \infty$ and $r \neq 1$; (d) $q = \infty$ and $r = 1$; (e) $1 = q \neq r$; (f) $1 \neq q = r$; (g) $1 = r \neq q$ and $q \neq 0, q \neq \infty$. Each case (a)–(g) is handled in a manner similar to the first two. In each case we obtain a contradiction; the details are routine and omitted. We have shown that A, B, C is isomorphic to at least one LR triple in Examples 28.1, 28.2. The result follows in view of Lemma 28.3(iii). \square

Lemma 28.5. Assume $d \geq 2$. Let A, B, C denote an LR triple in $\text{NBG}_d(\mathbb{F})$. Then for $0 \leq i \leq d-1$ the scalar ρ_i from Definition 17.7 satisfies

$$\frac{\text{case}}{\rho_i} \left| \begin{array}{cc} \text{NBG}_d(\mathbb{F}; q) & \text{NBG}_d(\mathbb{F}; 1) \\ q^{2i-d+1} & 1 \end{array} \right.$$

Proof. Compute $\rho_i = \varphi_{i+1}/\varphi_{d-i}$ using the data in Examples 28.1, 28.2. \square

29 The classification of LR triples in $\text{NBNG}_d(\mathbb{F})$

In this section we classify up to isomorphism the LR triples in $\text{NBNG}_d(\mathbb{F})$, for even $d \geq 4$. We first describe some examples.

Example 29.1. The LR triple $\text{NBNG}_d(\mathbb{F}; t)$ is over \mathbb{F} , diameter d , nonbipartite, normalized, and satisfies

$$\begin{aligned} d &\geq 4; & d &\text{ is even; } & 0 &\neq t \in \mathbb{F}; \\ t^i &\neq 1 \quad (1 \leq i \leq d/2); & t^{d+1} &\neq 1; \\ \varphi_i &= \begin{cases} t^{i/2} - 1 & \text{if } i \text{ is even;} \\ t^{(i-d-1)/2} - 1 & \text{if } i \text{ is odd} \end{cases} & (1 \leq i \leq d). \end{aligned}$$

Lemma 29.2. For the LR triples in Example 29.1, (i) they exist; (ii) they are contained in $\text{NBNG}_d(\mathbb{F})$; (iii) they are mutually nonisomorphic.

Proof. (i) Similar to the proof of Lemma 28.3(i), except that the sequences $\{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^d$ are now defined as follows: $\alpha_0 = 1, \beta_0 = 1$ and for $1 \leq i \leq d$,

$$\begin{aligned} \alpha_{i-2}/\alpha_i &= 1 - t^{i/2}, & \beta_{i-2}/\beta_i &= 1 - t^{-i/2} & (\text{if } i \text{ is even}), \\ \alpha_i &= \alpha_{i-1}, & \beta_i &= -\beta_{i-1} & (\text{if } i \text{ is odd}). \end{aligned}$$

(ii) Let A, B, C denote an LR triple listed in Example 29.1. By assumption A, B, C is over \mathbb{F} , diameter d , nonbipartite, and normalized. We check that the sequence $\{\rho_i\}_{i=0}^{d-1}$ from Definition 17.7 is not geometric. For $0 \leq i \leq d-1$ the scalar $\rho_i = \varphi_{i+1}/\varphi_{d-i}$ is equal to $-t^{(i-d)/2}$ (if i is even) and $-t^{(i+1)/2}$ (if i is odd). By this and since $t^{d+1} \neq 1$, the sequence $\{\rho_i\}_{i=0}^{d-1}$ is not geometric. By these comments A, B, C is contained in $\text{NBNG}_d(\mathbb{F})$.

(iii) Similar to the proof of Lemma 28.3(iii). \square

Theorem 29.3. Assume that d is even and at least 4. Then each LR triple in $\text{NBNG}_d(\mathbb{F})$ is isomorphic to a unique LR triple listed in Example 29.1.

Proof. Let A, B, C denote an LR triple in $\text{NBNG}_d(\mathbb{F})$, with parameter array (44) and Toeplitz data (54). Recall the sequence $\{\rho_i\}_{i=0}^{d-1}$ from Definition 17.7. This sequence is constrained by Proposition 25.7, so by Proposition 11.4 there exists $0 \neq t \in \mathbb{F}$ such that

$$\rho_i = \begin{cases} \rho_0 t^{i/2} & \text{if } i \text{ is even;} \\ \rho_0^{-1} t^{(i-d+1)/2} & \text{if } i \text{ is odd} \end{cases} \quad (0 \leq i \leq d-1). \quad (178)$$

By assumption $\{\rho_i\}_{i=0}^{d-1}$ is not geometric, so by Lemma 11.5(iv),

$$\rho_0^4 \neq t^{1-d}. \quad (179)$$

We claim that

$$t(\varphi_{i-2} - \varphi_{i-1}) = \varphi_i - \varphi_{i+1} \quad (2 \leq i \leq d). \quad (180)$$

To prove the claim, consider the scalars a, b, c from Definition 25.3. By Lemma 25.5 the 3-tuple (a, b, c) is a linear constraint for $\{\rho_i\}_{i=0}^{d-1}$ in the sense of Definition 11.7. Now using Definition 11.9 and Proposition 11.10(ii) we obtain $a = -tc$ and $b = 0$. The claim follows from this and Lemma 25.4. We show that $t \neq 1$. Suppose $t = 1$. By (180), for $2 \leq i \leq d$ we have

$$\varphi_{i-2} - \varphi_{i-1} = \varphi_i - \varphi_{i+1}. \quad (181)$$

Sum (181) over $i = 2, 3, \dots, d$ and use $\varphi_0 = 0 = \varphi_{d+1}$ to obtain $-\varphi_{d-1} = \varphi_2$. Set $i = d - 2$ in (128) and use (178) with $t = 1$ to find $\rho_0 = \varphi_{d-1}/\varphi_2 = -1$, which contradicts (179). We have shown $t \neq 1$. There exist $H, K, L \in \mathbb{F}$ such that for $i = 0, 1, 2$,

$$\varphi_i = \begin{cases} H + Kt^{i/2} & \text{if } i \text{ is even;} \\ H + Lt^{(i-d-1)/2} & \text{if } i \text{ is odd.} \end{cases} \quad (182)$$

By (180) and induction on i , (182) holds for $0 \leq i \leq d + 1$. Using $\varphi_0 = 0$ and $\varphi_{d+1} = 0$, we obtain $H + K = 0$ and $H + L = 0$. Now (182) becomes

$$\varphi_i = \begin{cases} H(1 - t^{i/2}) & \text{if } i \text{ is even;} \\ H(1 - t^{(i-d-1)/2}) & \text{if } i \text{ is odd} \end{cases} \quad (1 \leq i \leq d). \quad (183)$$

The scalars $\{\varphi_i\}_{i=1}^d$ are nonzero. Consequently $H \neq 0$, and $t^i \neq 1$ for $1 \leq i \leq d/2$. Evaluating $\rho_0 = \varphi_1/\varphi_d$ using (183) we obtain $\rho_0 = -t^{-d/2}$. Now (178) becomes

$$\rho_i = \begin{cases} -t^{(i-d)/2} & \text{if } i \text{ is even;} \\ -t^{(i+1)/2} & \text{if } i \text{ is odd} \end{cases} \quad (0 \leq i \leq d - 1), \quad (184)$$

and (179) becomes $t^{d+1} \neq 1$. We show $H = -1$. As in the proof of Theorem 28.4, there exists $q \in \mathbb{F} \cup \{\infty\}$ such that $q \neq -1$ and

$$\rho_i = \frac{q\varphi_i - (q+1)\varphi_{i+1} + \varphi_{i+2}}{q+1} \quad (0 \leq i \leq d - 1). \quad (185)$$

Evaluate the recursion (185) using (183), (184). For i even this gives

$$1 + H^{-1} = \frac{q+t}{q+1} t^{d/2}, \quad (186)$$

and for i odd this gives

$$1 + H^{-1} = \frac{q+t}{q+1} t^{-d/2-1}. \quad (187)$$

Combining (186), (187) we obtain

$$\frac{(q+t)(1-t^{d+1})}{q+1} = 0.$$

But $t^{d+1} \neq 1$, so $q = -t$, and therefore $H = -1$ in view of (186). Setting $H = -1$ in (183) we obtain

$$\varphi_i = \begin{cases} t^{i/2} - 1 & \text{if } i \text{ is even;} \\ t^{(i-d-1)/2} - 1 & \text{if } i \text{ is odd} \end{cases} \quad (1 \leq i \leq d).$$

We have met the requirements of Example 29.1, so A, B, C is isomorphic to $\text{NBNG}_d(\mathbb{F}; t)$. We have shown that A, B, C is isomorphic to at least one LR triple in Example 29.1. The result follows in view of Lemma 29.2(iii). \square

We record a fact from the proof of Theorem 29.3.

Lemma 29.4. *Assume that d is even and at least 4. Let A, B, C denote an LR triple in $\text{NBNG}_d(\mathbb{F})$. Then for $0 \leq i \leq d-1$ the scalar ρ_i from Definition 17.7 satisfies*

$$\rho_i = \begin{cases} -t^{(i-d)/2} & \text{if } i \text{ is even;} \\ -t^{(i+1)/2} & \text{if } i \text{ is odd.} \end{cases}$$

30 The classification of LR triples in $B_d(\mathbb{F})$

In this section we classify up to isomorphism the LR triples in $B_d(\mathbb{F})$, for even $d \geq 2$. We first describe some examples.

Example 30.1. The LR triple $B_d(\mathbb{F}; t, \rho_0, \rho'_0, \rho''_0)$ is over \mathbb{F} , diameter d , bipartite, normalized, and satisfies

$$\begin{aligned} d &\geq 4; & d &\text{ is even;} & 0 &\neq t \in \mathbb{F}; & t^i &\neq 1 \quad (1 \leq i \leq d/2); \\ \rho_0, \rho'_0, \rho''_0 &\in \mathbb{F}; & \rho_0 \rho'_0 \rho''_0 &= -t^{1-d/2}; \\ \varphi_i &= \begin{cases} \rho_0 \frac{1-t^{i/2}}{1-t} & \text{if } i \text{ is even;} \\ \frac{t}{\rho_0} \frac{1-t^{(i-d-1)/2}}{1-t} & \text{if } i \text{ is odd} \end{cases} & (1 \leq i \leq d); \\ \varphi'_i &= \begin{cases} \rho'_0 \frac{1-t^{i/2}}{1-t} & \text{if } i \text{ is even;} \\ \frac{t}{\rho'_0} \frac{1-t^{(i-d-1)/2}}{1-t} & \text{if } i \text{ is odd} \end{cases} & (1 \leq i \leq d); \\ \varphi''_i &= \begin{cases} \rho''_0 \frac{1-t^{i/2}}{1-t} & \text{if } i \text{ is even;} \\ \frac{t}{\rho''_0} \frac{1-t^{(i-d-1)/2}}{1-t} & \text{if } i \text{ is odd} \end{cases} & (1 \leq i \leq d). \end{aligned}$$

Example 30.2. The LR triple $B_d(\mathbb{F}; 1, \rho_0, \rho'_0, \rho''_0)$ is over \mathbb{F} , diameter d , bipartite, normalized, and satisfies

$$\begin{aligned} d &\geq 4; & d \text{ is even}; & \text{Char}(\mathbb{F}) \text{ is } 0 \text{ or greater than } d/2; \\ \rho_0, \rho'_0, \rho''_0 &\in \mathbb{F}; & \rho_0 \rho'_0 \rho''_0 &= -1; \\ \varphi_i &= \begin{cases} \frac{i\rho_0}{2} & \text{if } i \text{ is even;} \\ \frac{i-d-1}{2\rho_0} & \text{if } i \text{ is odd} \end{cases} & (1 \leq i \leq d); \\ \varphi'_i &= \begin{cases} \frac{i\rho'_0}{2} & \text{if } i \text{ is even;} \\ \frac{i-d-1}{2\rho'_0} & \text{if } i \text{ is odd} \end{cases} & (1 \leq i \leq d); \\ \varphi''_i &= \begin{cases} \frac{i\rho''_0}{2} & \text{if } i \text{ is even;} \\ \frac{i-d-1}{2\rho''_0} & \text{if } i \text{ is odd} \end{cases} & (1 \leq i \leq d). \end{aligned}$$

Example 30.3. The LR triple $B_2(\mathbb{F}; \rho_0, \rho'_0, \rho''_0)$ is over \mathbb{F} , diameter 2, bipartite, normalized, and satisfies

$$\begin{aligned} \rho_0, \rho'_0, \rho''_0 &\in \mathbb{F}; & \rho_0 \rho'_0 \rho''_0 &= -1; \\ \varphi_1 &= -1/\rho_0, & \varphi'_1 &= -1/\rho'_0, & \varphi''_1 &= -1/\rho''_0, \\ \varphi_2 &= \rho_0, & \varphi'_2 &= \rho'_0, & \varphi''_2 &= \rho''_0. \end{aligned}$$

Lemma 30.4. For the LR triples in Examples 30.1–30.3, (i) they exist; (ii) they are contained in $B_d(\mathbb{F})$; (iii) they are mutually nonisomorphic.

Proof. Without loss we may assume $d \geq 4$, since for $d = 2$ the result follows from Lemma 24.4.

(i) In Example 30.1 we see a parameter $t \in \mathbb{F}$. For Example 30.2 define $t = 1$. We proceed as in the proof of Lemma 28.3(i), except that now $a_i = 0$ for $0 \leq i \leq d$ and the sequences $\{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^d$ are defined as follows: $\alpha_0 = 1, \beta_0 = 1$ and for $1 \leq i \leq d$,

$$\begin{aligned} \alpha_{i-2}/\alpha_i &= \sum_{k=0}^{i/2-1} t^k & \beta_{i-2}/\beta_i &= - \sum_{k=0}^{i/2-1} t^{-k} & (\text{if } i \text{ is even}), \\ \alpha_i &= 0, & \beta_i &= 0 & (\text{if } i \text{ is odd}). \end{aligned}$$

(ii) By construction.

(iii) Similar to the proof of Lemma 28.3(iii). \square

Theorem 30.5. Assume that d is even and at least 2. Then each LR triple in $B_d(\mathbb{F})$ is isomorphic to a unique LR triple listed in Examples 30.1–30.3.

Proof. Without loss we may assume $d \geq 4$, since for $d = 2$ the result follows from Lemma 24.4. Let A, B, C denote an LR triple in $B_d(\mathbb{F})$, with parameter array (44) and Toeplitz data (54). Recall the sequences $\{\rho_i\}_{i=0}^{d-1}, \{\rho'_i\}_{i=0}^{d-1}, \{\rho''_i\}_{i=0}^{d-1}$ from Definition 17.15. These sequences are constrained by Proposition 25.12, and related to each other by Lemma 17.17(iii). So by Proposition 11.4, there exists $0 \neq t \in \mathbb{F}$ such that for $0 \leq i \leq d-1$,

$$\frac{\rho_i}{\rho_0} = \frac{\rho'_i}{\rho'_0} = \frac{\rho''_i}{\rho''_0} = t^{i/2} \quad \text{if } i \text{ is even}; \quad (188)$$

$$\rho_i \rho_0 = \rho'_i \rho'_0 = \rho''_i \rho''_0 = t^{(i-d+1)/2} \quad \text{if } i \text{ is odd}. \quad (189)$$

We now compute $\{\varphi_i\}_{i=1}^d$. By Lemma 25.9 and since A, B, C is normalized,

$$\rho_i = \varphi_{i+2} - \varphi_i \quad (0 \leq i \leq d-1).$$

By this and since $\varphi_0 = 0 = \varphi_{d+1}$,

$$\varphi_i = \begin{cases} \rho_0 + \rho_2 + \rho_4 + \cdots + \rho_{i-2} & \text{if } i \text{ is even;} \\ -\rho_i - \rho_{i+2} - \rho_{i+4} - \cdots - \rho_{d-1} & \text{if } i \text{ is odd} \end{cases} \quad (1 \leq i \leq d). \quad (190)$$

Evaluate (190) using (188), (189) to obtain the formula for $\{\varphi_i\}_{i=1}^d$ given in Example 30.1 (if $t \neq 1$) or Example 30.2 (if $t = 1$). We similarly obtain the formulae for $\{\varphi'_i\}_{i=1}^d$, $\{\varphi''_i\}_{i=1}^d$ given in Examples 30.1, 30.2. Using these formulae and $\rho_0 = \varphi'_1/\varphi''_d$ we obtain the formula for $\rho_0\rho'_0\rho''_0$ given in Examples 30.1, 30.2. For the moment assume that $t \neq 1$. Then for $1 \leq i \leq d/2$, $t^i \neq 1$ since $\varphi_{2i} \neq 0$. We have met the requirements of Example 30.1, so A, B, C is isomorphic to $B_d(\mathbb{F}; t, \rho_0, \rho'_0, \rho''_0)$. Next assume that $t = 1$. Then for $1 \leq i \leq d/2$, $i \neq 0$ in \mathbb{F} since $\varphi_{2i} \neq 0$. Therefore $\text{Char}(\mathbb{F})$ is 0 or greater than $d/2$. We have met the requirements of Example 30.2, so A, B, C is isomorphic to $B_d(\mathbb{F}; 1, \rho_0, \rho'_0, \rho''_0)$. In summary, we have shown that A, B, C is isomorphic to at least one LR triple in Examples 30.1, 30.2. The result follows in view of Lemma 30.4(iii). \square

We record a result from the proof of Theorem 30.5.

Lemma 30.6. *Assume that d is even and at least 2. Let A, B, C denote an LR triple in $B_d(\mathbb{F})$. Then for $0 \leq i \leq d-1$ the scalars $\rho_i, \rho'_i, \rho''_i$ from Definition 17.15 are described as follows. For $B_d(\mathbb{F}; t, \rho_0, \rho'_0, \rho''_0)$,*

$$\begin{aligned} \rho_i &= \begin{cases} \rho_0 t^{i/2} & \text{if } i \text{ is even;} \\ \rho_0^{-1} t^{(i-d+1)/2} & \text{if } i \text{ is odd;} \end{cases} \\ \rho'_i &= \begin{cases} \rho'_0 t^{i/2} & \text{if } i \text{ is even;} \\ (\rho'_0)^{-1} t^{(i-d+1)/2} & \text{if } i \text{ is odd;} \end{cases} \\ \rho''_i &= \begin{cases} \rho''_0 t^{i/2} & \text{if } i \text{ is even;} \\ (\rho''_0)^{-1} t^{(i-d+1)/2} & \text{if } i \text{ is odd.} \end{cases} \end{aligned}$$

For $B_d(\mathbb{F}; 1, \rho_0, \rho'_0, \rho''_0)$ and $B_2(\mathbb{F}; \rho_0, \rho'_0, \rho''_0)$,

$$\begin{aligned} \rho_i &= \begin{cases} \rho_0 & \text{if } i \text{ is even;} \\ \rho_0^{-1} & \text{if } i \text{ is odd;} \end{cases} \\ \rho'_i &= \begin{cases} \rho'_0 & \text{if } i \text{ is even;} \\ (\rho'_0)^{-1} & \text{if } i \text{ is odd;} \end{cases} \\ \rho''_i &= \begin{cases} \rho''_0 & \text{if } i \text{ is even;} \\ (\rho''_0)^{-1} & \text{if } i \text{ is odd.} \end{cases} \end{aligned}$$

Proof. For $d \geq 4$ use (188), (189). For $d = 2$ use Definition 17.15 and Example 30.3. \square

We have now classified up to isomorphism the normalized LR triples over \mathbb{F} that have diameter $d \geq 2$.

Recall the similarity relation for LR triples, from Definition 13.12.

Corollary 30.7. *Consider the set of LR triples consisting of the LR triple in Lemma 24.2 and the LR triples in Examples 27.1–27.3, 28.1, 28.2, 29.1. Each nonbipartite LR triple over \mathbb{F} is similar to a unique LR triple in this set.*

Proof. By Definition 13.12, Corollary 18.7, Lemma 24.2, and Theorems 27.5, 28.4, 29.3. \square

Recall the bisimilarity relation for bipartite LR triples, from Definition 16.36.

Corollary 30.8. *Consider the set of LR triples consisting of Examples 30.1–30.3. Each nontrivial bipartite LR triple over \mathbb{F} is bisimilar to a unique LR triple in this set.*

Proof. By Definition 16.36, Corollary 18.16, and Theorem 30.5. \square

31 The Toeplitz data and unipotent maps

Throughout this section the following notation is in effect. Let V denote a vector space over \mathbb{F} with dimension $d + 1$. Let A, B, C denote an equitable LR triple on V , with parameter array (44) and Toeplitz data (54). By Definition 17.1 we have $\alpha_i = \alpha'_i = \alpha''_i$ for $0 \leq i \leq d$, and by Lemma 17.3 we have $\beta_i = \beta'_i = \beta''_i$ for $0 \leq i \leq d$. Recall the unipotent maps $\mathbb{A}, \mathbb{B}, \mathbb{C}$ from Definition 21.1. By Proposition 21.13,

$$\mathbb{A} = \sum_{i=0}^d \alpha_i A^i, \quad \mathbb{B} = \sum_{i=0}^d \alpha_i B^i, \quad \mathbb{C} = \sum_{i=0}^d \alpha_i C^i, \quad (191)$$

$$\mathbb{A}^{-1} = \sum_{i=0}^d \beta_i A^i, \quad \mathbb{B}^{-1} = \sum_{i=0}^d \beta_i B^i, \quad \mathbb{C}^{-1} = \sum_{i=0}^d \beta_i C^i. \quad (192)$$

Since A, B, C are Nil,

$$A^{d+1} = 0, \quad B^{d+1} = 0, \quad C^{d+1} = 0. \quad (193)$$

In this section we compute $\{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^d$ for the cases $\text{NBG}_d(\mathbb{F}), \text{NBNG}_d(\mathbb{F}), \text{B}_d(\mathbb{F})$. In each case, we relate $\mathbb{A}, \mathbb{B}, \mathbb{C}$ to the exponential function or quantum exponential function. We now recall these functions. In what follows, λ denotes an indeterminate. The infinite series that we will encounter should be viewed as formal sums; their convergence is not an issue.

Definition 31.1. Define

$$\exp(\lambda) = \sum_{i=0}^N \frac{\lambda^i}{i!},$$

where $N = \infty$ if $\text{Char}(\mathbb{F}) = 0$, and $N + 1 = \text{Char}(\mathbb{F})$ if $\text{Char}(\mathbb{F}) > 0$.

Definition 31.2. For a nonzero $q \in \mathbb{F}$ such that $q \neq 1$, define

$$\exp_q(\lambda) = \sum_{i=0}^N \frac{\lambda^i}{(1)(1+q)(1+q+q^2) \cdots (1+q+q^2+\cdots+q^{i-1})},$$

where $N = \infty$ if q is not a root of unity, and otherwise q is a primitive $(N+1)$ -root of unity.

Proposition 31.3. Assume that A, B, C is in $\text{NBG}_d(\mathbb{F})$ or $\text{NBNG}_d(\mathbb{F})$ or $\text{B}_d(\mathbb{F})$. Then the scalars $\{\alpha_i\}_{i=0}^d$, $\{\beta_i\}_{i=0}^d$ and maps \mathbb{A} , \mathbb{B} , \mathbb{C} are described as follows.

Case $\text{NBG}_d(\mathbb{F}; q)$. For $0 \leq i \leq d$,

$$\begin{aligned} \alpha_i &= \frac{1}{(1)(1+q)(1+q+q^2) \cdots (1+q+q^2+\cdots+q^{i-1})}, \\ \beta_i &= \frac{(-1)^i q^{\binom{i}{2}}}{(1)(1+q)(1+q+q^2) \cdots (1+q+q^2+\cdots+q^{i-1})} \\ &= \frac{(-1)^i}{(1)(1+q^{-1})(1+q^{-1}+q^{-2}) \cdots (1+q^{-1}+q^{-2}+\cdots+q^{1-i})}. \end{aligned}$$

Moreover,

$$\begin{aligned} \mathbb{A} &= \exp_q(A), & \mathbb{B} &= \exp_q(B), & \mathbb{C} &= \exp_q(C), \\ \mathbb{A}^{-1} &= \exp_{q^{-1}}(-A), & \mathbb{B}^{-1} &= \exp_{q^{-1}}(-B), & \mathbb{C}^{-1} &= \exp_{q^{-1}}(-C). \end{aligned}$$

Case $\text{NBG}_d(\mathbb{F}; 1)$. For $0 \leq i \leq d$,

$$\alpha_i = \frac{1}{i!}, \quad \beta_i = \frac{(-1)^i}{i!}.$$

Moreover,

$$\begin{aligned} \mathbb{A} &= \exp(A), & \mathbb{B} &= \exp(B), & \mathbb{C} &= \exp(C), \\ \mathbb{A}^{-1} &= \exp(-A), & \mathbb{B}^{-1} &= \exp(-B), & \mathbb{C}^{-1} &= \exp(-C). \end{aligned}$$

Case $\text{NBNG}_d(\mathbb{F}; t)$. For $0 \leq i \leq d/2$,

$$\begin{aligned} \alpha_{2i} &= \frac{1}{(1-t)(1-t^2) \cdots (1-t^i)}, \\ \beta_{2i} &= \frac{(-1)^i t^{\binom{i+1}{2}}}{(1-t)(1-t^2) \cdots (1-t^i)} \\ &= \frac{1}{(1-t^{-1})(1-t^{-2}) \cdots (1-t^{-i})}. \end{aligned}$$

For $0 \leq i \leq d/2 - 1$,

$$\alpha_{2i+1} = \alpha_{2i}, \quad \beta_{2i+1} = -\beta_{2i}.$$

Moreover,

$$\begin{aligned}\mathbb{A} &= (I + A)\exp_t\left(\frac{A^2}{1-t}\right), & \mathbb{A}^{-1} &= (I - A)\exp_{t^{-1}}\left(\frac{A^2}{1-t^{-1}}\right), \\ \mathbb{B} &= (I + B)\exp_t\left(\frac{B^2}{1-t}\right), & \mathbb{B}^{-1} &= (I - B)\exp_{t^{-1}}\left(\frac{B^2}{1-t^{-1}}\right), \\ \mathbb{C} &= (I + C)\exp_t\left(\frac{C^2}{1-t}\right), & \mathbb{C}^{-1} &= (I - C)\exp_{t^{-1}}\left(\frac{C^2}{1-t^{-1}}\right).\end{aligned}$$

Case $B_d(\mathbb{F}; t, \rho_0, \rho'_0, \rho''_0)$. For $0 \leq i \leq d/2$,

$$\begin{aligned}\alpha_{2i} &= \frac{1}{(1)(1+t)(1+t+t^2)\cdots(1+t+t^2+\cdots+t^{i-1})}, \\ \beta_{2i} &= \frac{(-1)^i t^{\binom{i}{2}}}{(1)(1+t)(1+t+t^2)\cdots(1+t+t^2+\cdots+t^{i-1})} \\ &= \frac{(-1)^i}{(1)(1+t^{-1})(1+t^{-1}+t^{-2})\cdots(1+t^{-1}+t^{-2}+\cdots+t^{1-i})}.\end{aligned}$$

For $0 \leq i \leq d/2 - 1$,

$$\alpha_{2i+1} = 0, \quad \beta_{2i+1} = 0.$$

Moreover,

$$\begin{aligned}\mathbb{A} &= \exp_t(A^2), & \mathbb{B} &= \exp_t(B^2), & \mathbb{C} &= \exp_t(C^2), \\ \mathbb{A}^{-1} &= \exp_{t^{-1}}(-A^2), & \mathbb{B}^{-1} &= \exp_{t^{-1}}(-B^2), & \mathbb{C}^{-1} &= \exp_{t^{-1}}(-C^2).\end{aligned}$$

Case $B_d(\mathbb{F}; 1, \rho_0, \rho'_0, \rho''_0)$. For $0 \leq i \leq d/2$,

$$\alpha_{2i} = \frac{1}{i!}, \quad \beta_{2i} = \frac{(-1)^i}{i!}.$$

For $0 \leq i \leq d/2 - 1$,

$$\alpha_{2i+1} = 0, \quad \beta_{2i+1} = 0.$$

Moreover,

$$\begin{aligned}\mathbb{A} &= \exp(A^2), & \mathbb{B} &= \exp(B^2), & \mathbb{C} &= \exp(C^2), \\ \mathbb{A}^{-1} &= \exp(-A^2), & \mathbb{B}^{-1} &= \exp(-B^2), & \mathbb{C}^{-1} &= \exp(-C^2).\end{aligned}$$

Case $B_2(\mathbb{F}; \rho_0, \rho'_0, \rho''_0)$. Same as $B_d(\mathbb{F}; 1; \rho_0, \rho'_0, \rho''_0)$ with $d = 2$.

Proof. Compute $\{\alpha_i\}_{i=0}^d, \{\beta_i\}_{i=0}^d$ as follows. In each nonbipartite case, use Proposition 15.14 and induction, together with

$$\alpha_0 = 1, \quad \alpha_1 = 1, \quad \beta_0 = 1, \quad \beta_1 = -1.$$

In each bipartite case, use Proposition 15.12 and induction, together with

$$\alpha_0 = 1, \quad \alpha_1 = 0, \quad \alpha_2 = 1, \quad \beta_0 = 1, \quad \beta_1 = 0, \quad \beta_2 = -1.$$

Our assertions about $\mathbb{A}, \mathbb{B}, \mathbb{C}$ follow from (191)–(193) and Definitions 31.1, 31.2. \square

32 Relations for LR triples

In this section we consider the relations satisfied by an LR triple A, B, C . In order to motivate our results, assume for the moment that A, B, C has q -Weyl type, in the sense of Definition 15.24. Then A, B, C satisfy the relations in (104) and Lemma 15.30. Next assume that A, B, C is contained in $\text{NBG}_d(\mathbb{F})$ or $\text{NBNG}_d(\mathbb{F})$ or $\text{B}_d(\mathbb{F})$. We show that A, B, C satisfy some analogous relations. We treat the nonbipartite cases $\text{NBG}_d(\mathbb{F})$, $\text{NBNG}_d(\mathbb{F})$ and the bipartite case $\text{B}_d(\mathbb{F})$ separately.

Proposition 32.1. *Let A, B, C denote an LR triple in $\text{NBG}_d(\mathbb{F})$ or $\text{NBNG}_d(\mathbb{F})$. Then A, B, C satisfy the following relations.*

Case $\text{NBG}_d(\mathbb{F}; q)$. *We have*

$$\begin{aligned} A^2B - q(1+q)ABA + q^3BA^2 &= q(1+q)A, \\ B^2C - q(1+q)BCB + q^3CB^2 &= q(1+q)B, \\ C^2A - q(1+q)CAC + q^3AC^2 &= q(1+q)C \end{aligned}$$

and also

$$\begin{aligned} AB^2 - q(1+q)BAB + q^3B^2A &= q(1+q)B, \\ BC^2 - q(1+q)CBC + q^3C^2B &= q(1+q)C, \\ CA^2 - q(1+q)ACA + q^3A^2C &= q(1+q)A. \end{aligned}$$

We have

$$\begin{aligned} A(I + (BC - qCB)(1 - q^{-1})) &= qB + q^{-1}C + qCB - q^{-1}BC, \\ B(I + (CA - qAC)(1 - q^{-1})) &= qC + q^{-1}A + qAC - q^{-1}CA, \\ C(I + (AB - qBA)(1 - q^{-1})) &= qA + q^{-1}B + qBA - q^{-1}AB \end{aligned}$$

and also

$$\begin{aligned} (I + (BC - qCB)(1 - q^{-1}))A &= q^{-1}B + qC + qCB - q^{-1}BC, \\ (I + (CA - qAC)(1 - q^{-1}))B &= q^{-1}C + qA + qAC - q^{-1}CA, \\ (I + (AB - qBA)(1 - q^{-1}))C &= q^{-1}A + qB + qBA - q^{-1}AB. \end{aligned}$$

We have

$$\begin{aligned} ABC - BCA + q(CBA - ACB) &= (1+q)(B - C), \\ BCA - CAB + q(ACB - BAC) &= (1+q)(C - A), \\ CAB - ABC + q(BAC - CBA) &= (1+q)(A - B) \end{aligned}$$

and also

$$\begin{aligned} &(1 + 2q^{-1})(ABC + BCA + CAB) - (1 + 2q)(CBA + ACB + BAC) \\ &= (q - q^{-1})(A + B + C) - \frac{3(q^d - 1)(q^{d+2} - 1)}{q^d(q - 1)^2}I. \end{aligned}$$

Case $\text{NBG}_d(\mathbb{F}; 1)$. *We have*

$$\begin{aligned} [A, [A, B]] &= 2A, & [B, [B, A]] &= 2B, \\ [B, [B, C]] &= 2B, & [C, [C, B]] &= 2C, \\ [C, [C, A]] &= 2C, & [A, [A, C]] &= 2A \end{aligned}$$

and also

$$A = B + C - [B, C], \quad B = C + A - [C, A], \quad C = A + B - [A, B].$$

We have

$$\begin{aligned} [A, [B, C]] &= 2(B - C), \\ [B, [C, A]] &= 2(C - A), \\ [C, [A, B]] &= 2(A - B) \end{aligned}$$

and also

$$ABC + BCA + CAB - CBA - ACB - BAC = -d(d+2)I.$$

Case $\text{NBNG}_d(\mathbb{F}; t)$. *We have*

$$\begin{aligned} \frac{A^2B - tBA^2}{1-t} &= -A, & \frac{B^2C - tCB^2}{1-t} &= -B, & \frac{C^2A - tAC^2}{1-t} &= -C, \\ \frac{AB^2 - tB^2A}{1-t} &= -B, & \frac{BC^2 - tC^2B}{1-t} &= -C, & \frac{CA^2 - tA^2C}{1-t} &= -A \end{aligned}$$

and also

$$\begin{aligned} \frac{ABC - tCBA}{1-t} + A + C &= -\frac{(1-t^{-d/2})(1-t^{1+d/2})}{1-t} I, \\ \frac{BCA - tACB}{1-t} + B + A &= -\frac{(1-t^{-d/2})(1-t^{1+d/2})}{1-t} I, \\ \frac{CAB - tBAC}{1-t} + C + B &= -\frac{(1-t^{-d/2})(1-t^{1+d/2})}{1-t} I. \end{aligned}$$

Proof. To verify these relations, represent A, B, C by matrices, using for example the first row of the table in Proposition 13.39. \square

Remark 32.2. In [4, p. 308] G. Benkart and T. Roby introduce the concept of a down-up algebra. Consider an LR triple A, B, C from Proposition 32.1. By that proposition, any two of A, B, C satisfy the defining relations for a down-up algebra.

Let A, B, C denote an LR triple in $B_d(\mathbb{F})$, and consider its projector J from Definition 16.24. By Lemma 16.26 we have $J^2 = J$, and by Lemma 16.27,

$$A = JA + AJ, \quad B = JB + BJ, \quad C = JC + CJ.$$

Proposition 32.3. *Let A, B, C denote an LR triple in $B_d(\mathbb{F})$. Then A, B, C and its projector J satisfy the following relations.*

Case $B_d(\mathbb{F}; t, \rho_0, \rho'_0, \rho''_0)$. *We have*

$$\begin{aligned} \left(\rho_0 AB + \rho'_0 \rho''_0 BA - \frac{1 - t^{-d/2}}{1 - t} tI \right) J &= 0, \\ \left(\rho'_0 BC + \rho''_0 \rho_0 CB - \frac{1 - t^{-d/2}}{1 - t} tI \right) J &= 0, \\ \left(\rho''_0 CA + \rho_0 \rho'_0 AC - \frac{1 - t^{-d/2}}{1 - t} tI \right) J &= 0 \end{aligned}$$

and also

$$\begin{aligned} \left(\rho'_0 \rho''_0 AB + \rho_0 tBA - \frac{1 - t^{-1-d/2}}{1 - t} t^2 I \right) (I - J) &= 0, \\ \left(\rho''_0 \rho_0 BC + \rho'_0 tCB - \frac{1 - t^{-1-d/2}}{1 - t} t^2 I \right) (I - J) &= 0, \\ \left(\rho_0 \rho'_0 CA + \rho''_0 tAC - \frac{1 - t^{-1-d/2}}{1 - t} t^2 I \right) (I - J) &= 0. \end{aligned}$$

We have

$$\begin{aligned} (A^2 B - tBA^2 - (t/\rho_0)A)J &= 0, & J(A^2 B - tBA^2 - \rho_0 A) &= 0, \\ (B^2 C - tCB^2 - (t/\rho'_0)B)J &= 0, & J(B^2 C - tCB^2 - \rho'_0 B) &= 0, \\ (C^2 A - tAC^2 - (t/\rho''_0)C)J &= 0, & J(C^2 A - tAC^2 - \rho''_0 C) &= 0 \end{aligned}$$

and also

$$\begin{aligned} J(AB^2 - tB^2 A - (t/\rho_0)B) &= 0, & (AB^2 - tB^2 A - \rho_0 B)J &= 0, \\ J(BC^2 - tC^2 B - (t/\rho'_0)C) &= 0, & (BC^2 - tC^2 B - \rho'_0 C)J &= 0, \\ J(CA^2 - tA^2 C - (t/\rho''_0)A) &= 0, & (CA^2 - tA^2 C - \rho''_0 A)J &= 0. \end{aligned}$$

We have

$$\begin{aligned} A^3 B + A^2 BA - tABA^2 - tBA^3 &= (\rho_0 + t/\rho_0)A^2, \\ B^3 C + B^2 CB - tBCB^2 - tCB^3 &= (\rho'_0 + t/\rho'_0)B^2, \\ C^3 A + C^2 AC - tCAC^2 - tAC^3 &= (\rho''_0 + t/\rho''_0)C^2 \end{aligned}$$

and also

$$\begin{aligned} AB^3 + BAB^2 - tB^2 AB - tB^3 A &= (\rho_0 + t/\rho_0)B^2, \\ BC^3 + CBC^2 - tC^2 BC - tC^3 B &= (\rho'_0 + t/\rho'_0)C^2, \\ CA^3 + ACA^2 - tA^2 CA - tA^3 C &= (\rho''_0 + t/\rho''_0)A^2. \end{aligned}$$

We have

$$\begin{aligned} \left(ABC - \frac{tA - \rho_0 tB + \rho_0 \rho'_0 C}{\rho'_0(1-t)}\right)J &= 0, & \left(CBA - \frac{tA - \rho_0 B + \rho_0 \rho'_0 C}{\rho'_0(1-t)}\right)J &= 0, \\ \left(BCA - \frac{tB - \rho'_0 tC + \rho'_0 \rho''_0 A}{\rho''_0(1-t)}\right)J &= 0, & \left(ACB - \frac{tB - \rho'_0 C + \rho'_0 \rho''_0 A}{\rho''_0(1-t)}\right)J &= 0, \\ \left(CAB - \frac{tC - \rho''_0 tA + \rho''_0 \rho_0 B}{\rho_0(1-t)}\right)J &= 0, & \left(BAC - \frac{tC - \rho''_0 A + \rho''_0 \rho_0 B}{\rho_0(1-t)}\right)J &= 0 \end{aligned}$$

and also

$$\begin{aligned} J\left(ABC - \frac{\rho_0 \rho'_0 A - \rho'_0 tB + tC}{\rho_0(1-t)}\right) &= 0, & J\left(CBA - \frac{\rho_0 \rho'_0 A - \rho'_0 B + tC}{\rho_0(1-t)}\right) &= 0, \\ J\left(BCA - \frac{\rho'_0 \rho''_0 B - \rho''_0 tC + tA}{\rho'_0(1-t)}\right) &= 0, & J\left(ACB - \frac{\rho'_0 \rho''_0 B - \rho''_0 C + tA}{\rho'_0(1-t)}\right) &= 0, \\ J\left(CAB - \frac{\rho''_0 \rho_0 C - \rho_0 tA + tB}{\rho''_0(1-t)}\right) &= 0, & J\left(BAC - \frac{\rho''_0 \rho_0 C - \rho_0 A + tB}{\rho''_0(1-t)}\right) &= 0. \end{aligned}$$

We have

$$\begin{aligned} (ABC - CBA - (\rho_0/\rho'_0)B)J &= 0, & J(ABC - CBA - (\rho'_0/\rho_0)B) &= 0, \\ (BCA - ACB - (\rho'_0/\rho''_0)C)J &= 0, & J(BCA - ACB - (\rho''_0/\rho'_0)C) &= 0, \\ (CAB - BAC - (\rho''_0/\rho_0)A)J &= 0, & J(CAB - BAC - (\rho_0/\rho''_0)A) &= 0. \end{aligned}$$

Case $B_d(\mathbb{F}; 1, \rho_0, \rho'_0, \rho''_0)$. We have

$$\begin{aligned} (\rho_0 AB + \rho'_0 \rho''_0 BA + (d/2)I)J &= 0, \\ (\rho'_0 BC + \rho''_0 \rho_0 CB + (d/2)I)J &= 0, \\ (\rho''_0 CA + \rho_0 \rho'_0 AC + (d/2)I)J &= 0 \end{aligned}$$

and also

$$\begin{aligned} \left(\rho'_0 \rho''_0 AB + \rho_0 BA + \frac{d+2}{2}I\right)(I-J) &= 0, \\ \left(\rho''_0 \rho_0 BC + \rho'_0 CB + \frac{d+2}{2}I\right)(I-J) &= 0, \\ \left(\rho_0 \rho'_0 CA + \rho''_0 AC + \frac{d+2}{2}I\right)(I-J) &= 0. \end{aligned}$$

We have

$$\begin{aligned} (A^2 B - BA^2 - A/\rho_0)J &= 0, & J(A^2 B - BA^2 - \rho_0 A) &= 0, \\ (B^2 C - CB^2 - B/\rho'_0)J &= 0, & J(B^2 C - CB^2 - \rho'_0 B) &= 0, \\ (C^2 A - AC^2 - C/\rho''_0)J &= 0, & J(C^2 A - AC^2 - \rho''_0 C) &= 0 \end{aligned}$$

and also

$$\begin{aligned} J(AB^2 - B^2 A - B/\rho_0) &= 0, & (AB^2 - B^2 A - \rho_0 B)J &= 0, \\ J(BC^2 - C^2 B - C/\rho'_0) &= 0, & (BC^2 - C^2 B - \rho'_0 C)J &= 0, \\ J(CA^2 - A^2 C - A/\rho''_0) &= 0, & (CA^2 - A^2 C - \rho''_0 A)J &= 0. \end{aligned}$$

We have

$$\begin{aligned} A^3B + A^2BA - ABA^2 - BA^3 &= (\rho_0 + 1/\rho_0)A^2, \\ B^3C + B^2CB - BCB^2 - CB^3 &= (\rho'_0 + 1/\rho'_0)B^2, \\ C^3A + C^2AC - CAC^2 - AC^3 &= (\rho''_0 + 1/\rho''_0)C^2 \end{aligned}$$

and also

$$\begin{aligned} AB^3 + BAB^2 - B^2AB - B^3A &= (\rho_0 + 1/\rho_0)B^2, \\ BC^3 + CBC^2 - C^2BC - C^3B &= (\rho'_0 + 1/\rho'_0)C^2, \\ CA^3 + ACA^2 - A^2CA - A^3C &= (\rho''_0 + 1/\rho''_0)A^2. \end{aligned}$$

We have

$$\begin{aligned} (A - B\rho_0 - C/\rho''_0)J &= 0, & J(A - B/\rho_0 - C\rho''_0) &= 0, \\ (B - C\rho'_0 - A/\rho_0)J &= 0, & J(B - C/\rho'_0 - A\rho_0) &= 0, \\ (C - A\rho''_0 - B/\rho'_0)J &= 0, & J(C - A/\rho''_0 - B\rho'_0) &= 0. \end{aligned}$$

Case $B_2(\mathbb{F}; \rho_0, \rho'_0, \rho''_0)$. Same as $B_d(\mathbb{F}; 1, \rho_0, \rho'_0, \rho''_0)$ with $d = 2$, where we interpret $d/2 = 1$ and $(d + 2)/2 = 0$ if $\text{Char}(\mathbb{F}) = 2$.

Proof. To verify these relations, represent A, B, C, J by matrices, using for example the first row of the table in Proposition 13.39, along with Lemma 16.28. \square

33 The quantum algebra $U_q(\mathfrak{sl}_2)$ and the Lie algebra \mathfrak{sl}_2

In this section, we discuss how LR triples are related to the quantum algebra $U_q(\mathfrak{sl}_2)$ and the Lie algebra \mathfrak{sl}_2 .

Until further notice, assume that the field \mathbb{F} is arbitrary, and fix a nonzero $q \in \mathbb{F}$ such that $q^2 \neq 1$. We recall the algebra $U_q(\mathfrak{sl}_2)$. We will use the equitable presentation, which was introduced in [18].

Definition 33.1. (See [18, Theorem 2.1].) Let $U_q(\mathfrak{sl}_2)$ denote the \mathbb{F} -algebra with generators $x, y^{\pm 1}, z$ and relations $yy^{-1} = 1, y^{-1}y = 1$,

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \quad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \quad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1. \quad (194)$$

The following subalgebra of $U_q(\mathfrak{sl}_2)$ is of interest.

Definition 33.2. Let $U_q^R(\mathfrak{sl}_2)$ denote the subalgebra of $U_q(\mathfrak{sl}_2)$ generated by x, y, z . We call $U_q^R(\mathfrak{sl}_2)$ the *reduced* $U_q(\mathfrak{sl}_2)$ algebra.

Lemma 33.3. (See [22, Definition 10.6, Lemma 10.9].) The algebra $U_q^R(\mathfrak{sl}_2)$ has a presentation by generators x, y, z and relations

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \quad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \quad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1. \quad (195)$$

There is a central element Λ in $U_q(\mathfrak{sl}_2)$ that is often called the normalized Casimir element [22, Definition 2.11]. The element Λ is equal to $(q - q^{-1})^2$ times the Casimir element given in [19, Section 2.7]. By [22, Lemma 2.15], Λ is equal to each of the following:

$$qx + q^{-1}y + qz - qxyz, \quad q^{-1}x + qy + q^{-1}z - q^{-1}zyx, \quad (196)$$

$$qy + q^{-1}z + qx - qyzx, \quad q^{-1}y + qz + q^{-1}x - q^{-1}xzy, \quad (197)$$

$$qz + q^{-1}x + qy - qzxy, \quad q^{-1}z + qx + q^{-1}y - q^{-1}yxz. \quad (198)$$

Note that Λ is contained in $U_q^R(\mathfrak{sl}_2)$.

Recall from Definition 15.24 the LR triples of q -Weyl type.

Proposition 33.4. *Let A, B, C denote an LR triple over \mathbb{F} that has q -Weyl type. Then the underlying vector space V becomes a $U_q^R(\mathfrak{sl}_2)$ -module on which*

$$A = x, \quad B = y, \quad C = z. \quad (199)$$

The $U_q^R(\mathfrak{sl}_2)$ -module V is irreducible. On the $U_q^R(\mathfrak{sl}_2)$ -module V ,

$$\Lambda = \alpha_1(q - q^{-1})I, \quad (200)$$

where α_1 is the first Toeplitz number for A, B, C .

Proof. Compare (104), (195) to obtain the first assertion. The $U_q^R(\mathfrak{sl}_2)$ -module V is irreducible by (199) and Lemma 13.20. To get (200), compare Lemma 15.30 and (196)–(198) using (199). \square

We are done discussing the LR triples of q -Weyl type.

We return our attention to $U_q(\mathfrak{sl}_2)$. By [18, Lemma 5.1] we find that in $U_q(\mathfrak{sl}_2)$,

$$q(1 - xy) = q^{-1}(1 - yx), \quad q(1 - yz) = q^{-1}(1 - zy), \quad q(1 - zx) = q^{-1}(1 - xz).$$

Definition 33.5. (See [18, Definition 5.2].) Let n_x, n_y, n_z denote the following elements in $U_q(\mathfrak{sl}_2)$:

$$\begin{aligned} n_x &= \frac{q(1 - yz)}{q - q^{-1}} = \frac{q^{-1}(1 - zy)}{q - q^{-1}}, \\ n_y &= \frac{q(1 - zx)}{q - q^{-1}} = \frac{q^{-1}(1 - xz)}{q - q^{-1}}, \\ n_z &= \frac{q(1 - xy)}{q - q^{-1}} = \frac{q^{-1}(1 - yx)}{q - q^{-1}}. \end{aligned}$$

Lemma 33.6. (See [18, Lemma 5.4].) *The following relations hold in $U_q(\mathfrak{sl}_2)$:*

$$\begin{aligned} xn_y &= q^2 n_y x, & xn_z &= q^{-2} n_z x, \\ yn_z &= q^2 n_z y, & yn_x &= q^{-2} n_x y, \\ zn_x &= q^2 n_x z, & zn_y &= q^{-2} n_y z. \end{aligned}$$

Until further notice, let A, B, C denote an LR triple that is contained in $\text{NBG}_d(\mathbb{F}; q^{-2})$. Let V denote the underlying vector space.

Definition 33.7. Define X, Y, Z in $\text{End}(V)$ such that for $0 \leq i \leq d$, $X - q^{d-2i}I$ (resp. $Y - q^{d-2i}I$) (resp. $Z - q^{d-2i}I$) vanishes on component i of the (B, C) -decomposition (resp. (C, A) -decomposition) (resp. (A, B) -decomposition) of V . Note that each of X, Y, Z is invertible.

The next result is meant to clarify Definition 33.7. Recall the idempotent data (48) for A, B, C .

Lemma 33.8. *The elements X, Y, Z from Definition 33.7 satisfy*

$$X = \sum_{i=0}^d q^{d-2i} E'_i, \quad Y = \sum_{i=0}^d q^{d-2i} E''_i, \quad Z = \sum_{i=0}^d q^{d-2i} E_i.$$

Proof. By Definition 33.7 and the meaning of the idempotent data. \square

Lemma 33.9. *The elements X, Y, Z from Definition 33.7 satisfy*

$$\begin{aligned} X &= \frac{(q + q^{-1})I - (q - q^{-1})(q^2 BC - q^{-2} CB)}{q^{d+1} + q^{-d-1}}, \\ Y &= \frac{(q + q^{-1})I - (q - q^{-1})(q^2 CA - q^{-2} AC)}{q^{d+1} + q^{-d-1}}, \\ Z &= \frac{(q + q^{-1})I - (q - q^{-1})(q^2 AB - q^{-2} BA)}{q^{d+1} + q^{-d-1}}. \end{aligned}$$

Proof. To verify the first equation, work with the matrices in $\text{Mat}_{d+1}(\mathbb{F})$ that represent B, C, X with respect to a (B, C) -basis for V . For B, C these matrices are given in Proposition 13.39. For X this matrix is diagonal, with (i, i) -entry q^{d-2i} for $0 \leq i \leq d$. The other two equations are similarly verified. \square

Lemma 33.10. *The elements X, Y, Z from Definition 33.7 satisfy*

$$\frac{qXY - q^{-1}YX}{q - q^{-1}} = I, \quad \frac{qYZ - q^{-1}ZY}{q - q^{-1}} = I, \quad \frac{qZX - q^{-1}XZ}{q - q^{-1}} = I.$$

Proof. To verify these equations, eliminate X, Y, Z using Lemma 33.9, and evaluate the result using the relations for $\text{NBG}_d(\mathbb{F}; q^{-2})$ given in Proposition 32.1. \square

Proposition 33.11. *Let A, B, C denote an LR triple contained in $\text{NBG}_d(\mathbb{F}; q^{-2})$. Then there exists a unique $U_q(\mathfrak{sl}_2)$ -module structure on the underlying vector space V , such that for $0 \leq i \leq d$, $x - q^{d-2i}1$ (resp. $y - q^{d-2i}1$) (resp. $z - q^{d-2i}1$) vanishes on component i of the (B, C) -decomposition (resp. (C, A) -decomposition) (resp. (A, B) -decomposition) of V . The $U_q(\mathfrak{sl}_2)$ -module V is irreducible. On the $U_q(\mathfrak{sl}_2)$ -module V ,*

$$A = n_x, \quad B = n_y, \quad C = n_z. \quad (201)$$

Proof. The $U_q(\mathfrak{sl}_2)$ -module structure exists, by Lemma 33.10 and since Y is invertible. The $U_q(\mathfrak{sl}_2)$ -module structure is unique by construction. On the $U_q(\mathfrak{sl}_2)$ -module V we have $x = X$, $y = Y$, $z = Z$. To verify (201), eliminate n_x , n_y , n_z using Definition 33.5, and evaluate the result using Lemma 33.9 along with the relations for $\text{NBG}_d(\mathbb{F}; q^{-2})$ given in Proposition 32.1. The $U_q(\mathfrak{sl}_2)$ -module V is irreducible by (201) and Lemma 13.20. \square

We are done discussing the LR triples contained in $\text{NBG}_d(\mathbb{F}; q^{-2})$.

In a moment we will discuss the LR triples contained in $\text{NBNG}_d(\mathbb{F}; q^{-2})$. To prepare for this, we have some comments about $U_q(\mathfrak{sl}_2)$.

Lemma 33.12. *Assume that $q^4 \neq 1$. Then in $U_q(\mathfrak{sl}_2)$,*

$$\frac{q^2 x^2 y - q^{-2} y x^2}{q^2 - q^{-2}} = x, \quad \frac{q^2 y^2 z - q^{-2} z y^2}{q^2 - q^{-2}} = y, \quad \frac{q^2 z^2 x - q^{-2} x z^2}{q^2 - q^{-2}} = z. \quad (202)$$

Proof. We verify the equation on the left in (202). In the equation on the left in (194), multiply each term on the left by x to get

$$\frac{q x^2 y - q^{-1} x y x}{q - q^{-1}} = x. \quad (203)$$

Also, in the equation on the left in (194), multiply each term on the right by x to get

$$\frac{q x y x - q^{-1} y x^2}{q - q^{-1}} = x. \quad (204)$$

Now in (203), eliminate $x y x$ using (204) to obtain the equation on the left in (202). The remaining equations in (202) are similarly verified. \square

Lemma 33.13. *Assume that $q^4 \neq 1$. Then in $U_q(\mathfrak{sl}_2)$,*

$$\frac{q^2 x y^2 - q^{-2} y^2 x}{q^2 - q^{-2}} = y, \quad \frac{q^2 y z^2 - q^{-2} z^2 y}{q^2 - q^{-2}} = z, \quad \frac{q^2 z x^2 - q^{-2} x^2 z}{q^2 - q^{-2}} = x. \quad (205)$$

Proof. Similar to the proof of Lemma 33.12. \square

Lemma 33.14. *Assume that $q^4 \neq 1$. Then in $U_q(\mathfrak{sl}_2)$,*

$$-\frac{\Lambda}{q + q^{-1}} = \frac{q^2 x y z - q^{-2} z y x}{q^2 - q^{-2}} - x - z \quad (206)$$

$$= \frac{q^2 y z x - q^{-2} x z y}{q^2 - q^{-2}} - y - x \quad (207)$$

$$= \frac{q^2 z x y - q^{-2} y x z}{q^2 - q^{-2}} - z - y. \quad (208)$$

Proof. We verify (206). We have

$$\Lambda = q x + q^{-1} y + q z - q x y z,$$

so

$$q\Lambda = q^2x + y + q^2z - q^2xyz. \quad (209)$$

We have

$$\Lambda = q^{-1}x + qy + q^{-1}z - q^{-1}zyx,$$

so

$$q^{-1}\Lambda = q^{-2}x + y + q^{-2}z - q^{-2}zyx. \quad (210)$$

Subtract (210) from (209) and simplify to get (206). The equations (207), (208) are similarly verified. \square

Definition 33.15. Let $U_q^E(\mathfrak{sl}_2)$ denote the \mathbb{F} -algebra with generators x, y, z and relations

$$\begin{aligned} \frac{qx^2y - q^{-1}yx^2}{q - q^{-1}} &= -x, & \frac{qy^2z - q^{-1}zy^2}{q - q^{-1}} &= -y, & \frac{qz^2x - q^{-1}xz^2}{q - q^{-1}} &= -z, \\ \frac{qxy^2 - q^{-1}y^2x}{q - q^{-1}} &= -y, & \frac{qyz^2 - q^{-1}z^2y}{q - q^{-1}} &= -z, & \frac{qzx^2 - q^{-1}x^2z}{q - q^{-1}} &= -x, \\ \frac{qxyz - q^{-1}zyx}{q - q^{-1}} + x + z &= \frac{qyzx - q^{-1}xzy}{q - q^{-1}} + y + x = \frac{qzxy - q^{-1}yxz}{q - q^{-1}} + z + y. \end{aligned} \quad (211)$$

We call $U_q^E(\mathfrak{sl}_2)$ the *extended* $U_q(\mathfrak{sl}_2)$ algebra. Let Ω denote the common value of (211).

Lemma 33.16. *The element Ω from Definition 33.15 is central in $U_q^E(\mathfrak{sl}_2)$.*

Proof. Using the relations from Definition 33.15, one checks that Ω commutes with each generator x, y, z of $U_q^E(\mathfrak{sl}_2)$. \square

Lemma 33.17. *Assume that $q^4 \neq 1$ and there exists $i \in \mathbb{F}$ such that $i^2 = -1$. Then there exists an \mathbb{F} -algebra homomorphism $U_{q^2}^E(\mathfrak{sl}_2) \rightarrow U_q(\mathfrak{sl}_2)$ that sends*

$$x \mapsto ix, \quad y \mapsto iy, \quad z \mapsto iz, \quad \Omega \mapsto \frac{i\Lambda}{q + q^{-1}}.$$

Proof. Compare the relations in Lemmas 33.12–33.14 with the relations in Definition 33.15. \square

Proposition 33.18. *Let A, B, C denote an LR triple contained in $\text{NBNG}_d(\mathbb{F}; q^{-2})$. Then the underlying vector space V becomes a $U_q^E(\mathfrak{sl}_2)$ -module on which*

$$A = x, \quad B = y, \quad C = z. \quad (212)$$

The $U_q^E(\mathfrak{sl}_2)$ -module V is irreducible. On the $U_q^E(\mathfrak{sl}_2)$ -module V ,

$$\Omega = \frac{(q^{d/2} - q^{-d/2})(q^{1+d/2} - q^{-1-d/2})}{q - q^{-1}} I. \quad (213)$$

Proof. To get the first assertion and (213), compare the relations in Definition 33.15 with the relations for $\text{NBNG}_d(\mathbb{F}; q^{-2})$ given in Proposition 32.1. The $U_q^E(\mathfrak{sl}_2)$ -module V is irreducible by (212) and Lemma 13.20. \square

We are done discussing the LR triples contained in $\text{NBNG}_d(\mathbb{F}; q^{-2})$.

Until further notice let A, B, C denote an LR triple that is contained in $B_d(\mathbb{F}; q^{-2}, \rho_0, \rho'_0, \rho''_0)$. Let V denote the underlying vector space, and let J denote the projector.

Definition 33.19. Define X, Y, Z in $\text{End}(V)$ such that for $0 \leq i \leq d$, $X - q^{d/2-i}I$ (resp. $Y - q^{d/2-i}I$) (resp. $Z - q^{d/2-i}I$) vanishes on component i of the (B, C) -decomposition (resp. (C, A) -decomposition) (resp. (A, B) -decomposition) of V . Note that each of X, Y, Z is invertible.

Recall the idempotent data (48) for A, B, C .

Lemma 33.20. *The elements X, Y, Z from Definition 33.19 satisfy*

$$X = \sum_{i=0}^d q^{d/2-i} E'_i, \quad Y = \sum_{i=0}^d q^{d/2-i} E''_i, \quad Z = \sum_{i=0}^d q^{d/2-i} E_i.$$

Proof. By Definition 33.19 and the meaning of the idempotent data. \square

Lemma 33.21. *The elements X, Y, Z from Definition 33.19 satisfy*

$$\begin{aligned} X &= (q^{-d/2}I - BCq^{1-d/2}(q - q^{-1})\rho'_0)J + (q^{1+d/2}I - BCq^{d/2}(q - q^{-1})/\rho'_0)(I - J), \\ Y &= (q^{-d/2}I - CAq^{1-d/2}(q - q^{-1})\rho''_0)J + (q^{1+d/2}I - CAq^{d/2}(q - q^{-1})/\rho''_0)(I - J), \\ Z &= (q^{-d/2}I - ABq^{1-d/2}(q - q^{-1})\rho_0)J + (q^{1+d/2}I - ABq^{d/2}(q - q^{-1})/\rho_0)(I - J). \end{aligned}$$

Proof. To verify the first equation, work with the matrices in $\text{Mat}_{d+1}(\mathbb{F})$ that represent B, C, J, X with respect to a (B, C) -basis for V . For B, C these matrices are given in Proposition 13.39. For J this matrix is given in Lemma 16.28. For X this matrix is diagonal, with (i, i) -entry $q^{d/2-i}$ for $0 \leq i \leq d$. The other two equations are similarly verified. \square

Lemma 33.22. *The elements X, Y, Z from Definition 33.19 satisfy*

$$\frac{qXY - q^{-1}YX}{q - q^{-1}} = I, \quad \frac{qYZ - q^{-1}ZY}{q - q^{-1}} = I, \quad \frac{qZX - q^{-1}XZ}{q - q^{-1}} = I.$$

Proof. To verify these equations, eliminate X, Y, Z using Lemma 33.21, and evaluate the result using the relations for $B_d(\mathbb{F}; q^{-2}, \rho_0, \rho'_0, \rho''_0)$ given in Proposition 32.3. \square

Proposition 33.23. *Let A, B, C denote an LR triple contained in $B_d(\mathbb{F}; q^{-2}, \rho_0, \rho'_0, \rho''_0)$. Then there exists a unique $U_q(\mathfrak{sl}_2)$ -module structure on the underlying vector space V , such that for $0 \leq i \leq d$, $x - q^{d/2-i}1$ (resp. $y - q^{d/2-i}1$) (resp. $z - q^{d/2-i}1$) vanishes on component i of the (B, C) -decomposition (resp. (C, A) -decomposition) (resp. (A, B) -decomposition) of V . For the $U_q(\mathfrak{sl}_2)$ -module V the subspaces V_{out} and V_{in} are irreducible $U_q(\mathfrak{sl}_2)$ -submodules. On the $U_q(\mathfrak{sl}_2)$ -module V ,*

$$A^2 = n_x, \quad B^2 = n_y, \quad C^2 = n_z. \quad (214)$$

Proof. The $U_q(\mathfrak{sl}_2)$ -module structure exists, by Lemma 33.22 and since Y is invertible. The $U_q(\mathfrak{sl}_2)$ -module structure is unique by construction. On the $U_q(\mathfrak{sl}_2)$ -module V we have $x = X$, $y = Y$, $z = Z$. To verify (214), eliminate n_x, n_y, n_z using Definition 33.5, and evaluate the result using Lemma 33.21 along with the relations for $B_d(\mathbb{F}; q^{-2}, \rho_0, \rho'_0, \rho''_0)$ given in Proposition 32.3. By construction V_{out} and V_{in} are $U_q(\mathfrak{sl}_2)$ -submodules of V . By Lemmas 16.17, 16.18 the 3-tuple A^2, B^2, C^2 acts on V_{out} and V_{in} as an LR triple. By these comments along with (214) and Lemma 13.20, we find that the $U_q(\mathfrak{sl}_2)$ -submodules V_{out} and V_{in} are irreducible. \square

We mention some additional relations that hold on the $U_q(\mathfrak{sl}_2)$ -module V from Proposition 33.23. These relations may be of independent interest.

Lemma 33.24. *For the $U_q(\mathfrak{sl}_2)$ -module V from Proposition 33.23, we have*

$$\begin{aligned} xB &= qBx, & yC &= qCy, & zA &= qAz, \\ yA &= q^{-1}Ay, & zB &= q^{-1}Bz, & xC &= q^{-1}Cx \end{aligned}$$

and also

$$Jx = xJ, \quad Jy = yJ, \quad Jz = zJ.$$

We have

$$\begin{aligned} \left(AB - \frac{I - q^{d/2}z}{\rho_0 q(q - q^{-1})} \right) J &= 0, & \left(BA - \frac{\rho_0 q I - \rho_0 q^{1-d/2}z}{q - q^{-1}} \right) J &= 0, \\ \left(BC - \frac{I - q^{d/2}x}{\rho'_0 q(q - q^{-1})} \right) J &= 0, & \left(CB - \frac{\rho'_0 q I - \rho'_0 q^{1-d/2}x}{q - q^{-1}} \right) J &= 0, \\ \left(CA - \frac{I - q^{d/2}y}{\rho''_0 q(q - q^{-1})} \right) J &= 0, & \left(AC - \frac{\rho''_0 q I - \rho''_0 q^{1-d/2}y}{q - q^{-1}} \right) J &= 0 \end{aligned}$$

and also

$$\begin{aligned} \left(AB - \frac{\rho_0 q I - \rho_0 q^{-d/2}z}{q - q^{-1}} \right) (I - J) &= 0, & \left(BA - \frac{I - q^{1+d/2}z}{\rho_0 q(q - q^{-1})} \right) (I - J) &= 0, \\ \left(BC - \frac{\rho'_0 q I - \rho'_0 q^{-d/2}x}{q - q^{-1}} \right) (I - J) &= 0, & \left(CB - \frac{I - q^{1+d/2}x}{\rho'_0 q(q - q^{-1})} \right) (I - J) &= 0, \\ \left(CA - \frac{\rho''_0 q I - \rho''_0 q^{-d/2}y}{q - q^{-1}} \right) (I - J) &= 0, & \left(AC - \frac{I - q^{1+d/2}y}{\rho''_0 q(q - q^{-1})} \right) (I - J) &= 0. \end{aligned}$$

We have

$$\begin{aligned} (Ax - Bq^{-d/2}\rho_0 - Cq^{d/2}/\rho''_0)J &= 0, & (xA - Bq^{1-d/2}\rho_0 - Cq^{d/2-1}/\rho''_0)J &= 0, \\ (By - Cq^{-d/2}\rho'_0 - Aq^{d/2}/\rho_0)J &= 0, & (yB - Cq^{1-d/2}\rho'_0 - Aq^{d/2-1}/\rho_0)J &= 0, \\ (Cz - Aq^{-d/2}\rho''_0 - Bq^{d/2}/\rho'_0)J &= 0, & (zC - Aq^{1-d/2}\rho''_0 - Bq^{d/2-1}/\rho'_0)J &= 0 \end{aligned}$$

and also

$$\begin{aligned} J(Ax - Bq^{d/2-1}/\rho_0 - Cq^{1-d/2}\rho''_0) &= 0, & J(xA - Bq^{d/2}/\rho_0 - Cq^{-d/2}\rho''_0) &= 0, \\ J(By - Cq^{d/2-1}/\rho'_0 - Aq^{1-d/2}\rho_0) &= 0, & J(yB - Cq^{d/2}/\rho'_0 - Aq^{-d/2}\rho_0) &= 0, \\ J(Cz - Aq^{d/2-1}/\rho''_0 - Bq^{1-d/2}\rho'_0) &= 0, & J(zC - Aq^{d/2}/\rho''_0 - Bq^{-d/2}\rho'_0) &= 0. \end{aligned}$$

Proof. Similar to the proof of Lemma 33.22. \square

We are done discussing the LR triples contained in $B_d(\mathbb{F}; q^{-2}, \rho_0, \rho'_0, \rho''_0)$.

For the rest of this section, assume that $\text{Char}(\mathbb{F}) \neq 2$. We now recall the Lie algebra \mathfrak{sl}_2 and its equitable basis.

Definition 33.25. (See [12, Lemma 3.2].) Let \mathfrak{sl}_2 denote the Lie algebra over \mathbb{F} with basis x, y, z and Lie bracket

$$[x, y] = 2x + 2y, \quad [y, z] = 2y + 2z, \quad [z, x] = 2z + 2x. \quad (215)$$

Until further notice let A, B, C denote an LR triple that is contained in $\text{NBG}_d(\mathbb{F}; 1)$. Let V denote the underlying vector space.

Definition 33.26. Define X, Y, Z in $\text{End}(V)$ such that for $0 \leq i \leq d$, $X - (2i - d)I$ (resp. $Y - (2i - d)I$) (resp. $Z - (2i - d)I$) vanishes on component i of the (B, C) -decomposition (resp. (C, A) -decomposition) (resp. (A, B) -decomposition) of V .

Recall the idempotent data (48) for A, B, C .

Lemma 33.27. *The elements X, Y, Z from Definition 33.26 satisfy*

$$X = \sum_{i=0}^d (2i - d) E'_i, \quad Y = \sum_{i=0}^d (2i - d) E''_i, \quad Z = \sum_{i=0}^d (2i - d) E_i.$$

Proof. By Definition 33.26 and the meaning of the idempotent data. \square

Lemma 33.28. *The elements X, Y, Z from Definition 33.26 satisfy*

$$X = B + C - A, \quad Y = C + A - B, \quad Z = A + B - C.$$

Proof. To verify the first equation, work with the matrices in $\text{Mat}_{d+1}(\mathbb{F})$ that represent A, B, C, X with respect to a (B, C) -basis for V . For A, B, C these matrices are given in Proposition 13.39. For X this matrix is diagonal, with (i, i) -entry $2i - d$ for $0 \leq i \leq d$. The other two equations are similarly verified. \square

Lemma 33.29. *The elements X, Y, Z from Definition 33.26 satisfy*

$$[X, Y] = 2X + 2Y, \quad [Y, Z] = 2Y + 2Z, \quad [Z, X] = 2Z + 2X.$$

Proof. To verify these equations, eliminate X, Y, Z using Lemma 33.28, and evaluate the result using the relations for $\text{NBG}_d(\mathbb{F}; 1)$ given in Proposition 32.1. \square

Proposition 33.30. *Let A, B, C denote an LR triple contained in $\text{NBG}_d(\mathbb{F}; 1)$. Then there exists a unique \mathfrak{sl}_2 -module structure on the underlying vector space V , such that for $0 \leq i \leq d$, $x - (2i - d)1$ (resp. $y - (2i - d)1$) (resp. $z - (2i - d)1$) vanishes on component i of the (B, C) -decomposition (resp. (C, A) -decomposition) (resp. (A, B) -decomposition) of V . The \mathfrak{sl}_2 -module V is irreducible. On the \mathfrak{sl}_2 -module V ,*

$$A = (y + z)/2, \quad B = (z + x)/2, \quad C = (x + y)/2. \quad (216)$$

Proof. The \mathfrak{sl}_2 -module structure exists by Lemma 33.29. The \mathfrak{sl}_2 -module structure is unique by construction. On the \mathfrak{sl}_2 -module V we have $x = X$, $y = Y$, $z = Z$. To verify (216), eliminate x, y, z using Lemma 33.28, and evaluate the result using the relations for $\text{NBG}_d(\mathbb{F}; 1)$ given in Proposition 32.1. The \mathfrak{sl}_2 -module V is irreducible by (216) and Lemma 13.20. \square

We are done discussing the LR triples contained in $\text{NBG}_d(\mathbb{F}; 1)$.

For the rest of this section let A, B, C denote an LR triple that is contained in $\text{B}_d(\mathbb{F}; 1, \rho_0, \rho'_0, \rho''_0)$. Let V denote the underlying vector space, and let J denote the projector.

Definition 33.31. Define X, Y, Z in $\text{End}(V)$ such that for $0 \leq i \leq d$, $X - (i - d/2)I$ (resp. $Y - (i - d/2)I$) (resp. $Z - (i - d/2)I$) vanishes on component i of the (B, C) -decomposition (resp. (C, A) -decomposition) (resp. (A, B) -decomposition) of V .

Recall the idempotent data (48) for A, B, C .

Lemma 33.32. *The elements X, Y, Z from Definition 33.31 satisfy*

$$X = \sum_{i=0}^d (i - d/2) E'_i, \quad Y = \sum_{i=0}^d (i - d/2) E''_i, \quad Z = \sum_{i=0}^d (i - d/2) E_i.$$

Proof. By Definition 33.31 and the meaning of the idempotent data. \square

Lemma 33.33. *The elements X, Y, Z from Definition 33.31 satisfy*

$$\begin{aligned} X &= \left(2\rho'_0 BC + \frac{d}{2}I\right)J + \left(\frac{2}{\rho'_0}BC - \frac{d+2}{2}I\right)(I - J), \\ Y &= \left(2\rho''_0 CA + \frac{d}{2}I\right)J + \left(\frac{2}{\rho''_0}CA - \frac{d+2}{2}I\right)(I - J), \\ Z &= \left(2\rho_0 AB + \frac{d}{2}I\right)J + \left(\frac{2}{\rho_0}AB - \frac{d+2}{2}I\right)(I - J). \end{aligned}$$

Proof. To verify the first equation, work with the matrices in $\text{Mat}_{d+1}(\mathbb{F})$ that represent B, C, J, X with respect to a (B, C) -basis for V . For B, C these matrices are given in Proposition 13.39. For J this matrix is given in Lemma 16.28. For X this matrix is diagonal, with (i, i) -entry $i - d/2$ for $0 \leq i \leq d$. The other two equations are similarly verified. \square

Lemma 33.34. *The elements X, Y, Z from Definition 33.31 satisfy*

$$[X, Y] = 2X + 2Y, \quad [Y, Z] = 2Y + 2Z, \quad [Z, X] = 2Z + 2X.$$

Proof. To verify these equations, eliminate X, Y, Z using Lemma 33.33, and evaluate the result using the relations for $\text{B}_d(\mathbb{F}; 1, \rho_0, \rho'_0, \rho''_0)$ given in Proposition 32.3. \square

Proposition 33.35. *Let A, B, C denote an LR triple contained in $\text{B}_d(\mathbb{F}; 1, \rho_0, \rho'_0, \rho''_0)$. Then there exists a unique \mathfrak{sl}_2 -module structure on the underlying vector space V , such that for $0 \leq i \leq d$, $x - (i - d/2)1$ (resp. $y - (i - d/2)1$) (resp. $z - (i - d/2)1$) vanishes on component i of the (B, C) -decomposition (resp. (C, A) -decomposition) (resp. (A, B) -decomposition) of V . For the \mathfrak{sl}_2 -module V the subspaces V_{out} and V_{in} are irreducible \mathfrak{sl}_2 -submodules. On the \mathfrak{sl}_2 -module V ,*

$$A^2 = (y + z)/2, \quad B^2 = (z + x)/2, \quad C^2 = (x + y)/2. \quad (217)$$

Proof. Similar to the proof of Proposition 33.23. \square

We mention some additional relations that hold on the \mathfrak{sl}_2 -module V from Proposition 33.35. These relations may be of independent interest.

Lemma 33.36. *For the \mathfrak{sl}_2 -module V from Proposition 33.35, we have*

$$[A, z] = A, \quad [B, x] = B, \quad [C, y] = C$$

and also

$$[J, x] = 0, \quad [J, y] = 0, \quad [J, z] = 0.$$

We have

$$\begin{aligned} \left(AB - \frac{2z-d}{4\rho_0}\right)J &= 0, & \left(BA - \frac{\rho_0(2z+d)}{4}\right)J &= 0, \\ \left(BC - \frac{2x-d}{4\rho'_0}\right)J &= 0, & \left(CB - \frac{\rho'_0(2x+d)}{4}\right)J &= 0, \\ \left(CA - \frac{2y-d}{4\rho''_0}\right)J &= 0, & \left(AC - \frac{\rho''_0(2y+d)}{4}\right)J &= 0 \end{aligned}$$

and also

$$\begin{aligned} \left(AB - \frac{\rho_0(2z+d+2)}{4}\right)(I-J) &= 0, & \left(BA - \frac{2z-d-2}{4\rho_0}\right)(I-J) &= 0, \\ \left(BC - \frac{\rho'_0(2x+d+2)}{4}\right)(I-J) &= 0, & \left(CB - \frac{2x-d-2}{4\rho'_0}\right)(I-J) &= 0, \\ \left(CA - \frac{\rho''_0(2y+d+2)}{4}\right)(I-J) &= 0, & \left(AC - \frac{2y-d-2}{4\rho''_0}\right)(I-J) &= 0. \end{aligned}$$

Proof. Similar to the proof of Lemma 33.34. \square

34 Three characterizations of an LR triple

Throughout this section let V denote a vector space over \mathbb{F} with dimension $d+1$. We characterize the LR triples on V in three ways.

A matrix $M \in \text{Mat}_{d+1}(\mathbb{F})$ will be called *antidiagonal* whenever the entry $M_{i,j} = 0$ if $i+j \neq d$ ($0 \leq i, j \leq d$). Note that the following are equivalent: (i) M is antidiagonal; (ii) $\mathbf{Z}M$ is diagonal; (iii) $M\mathbf{Z}$ is diagonal.

Theorem 34.1. *Suppose we are given six bases for V , denoted*

$$\text{basis 1,} \quad \text{basis 2,} \quad \text{basis 3,} \quad \text{basis 4,} \quad \text{basis 5,} \quad \text{basis 6.} \quad (218)$$

Then the following are equivalent:

(i) *The transition matrix*

from basis	to basis	is
1	2	upper triangular Toeplitz
2	3	antidiagonal
3	4	upper triangular Toeplitz
4	5	antidiagonal
5	6	upper triangular Toeplitz
6	1	antidiagonal

(ii) *There exists an LR triple A, B, C on V for which*

basis	has type
1	(A, C)
2	(A, B)
3	(B, A)
4	(B, C)
5	(C, B)
6	(C, A)

Suppose (i), (ii) hold. Then the LR triple A, B, C is uniquely determined by the sequence (218).

Proof. (i) \Rightarrow (ii) For $1 \leq k \leq 6$ let \mathcal{D}_k denote the decomposition of V induced by basis k . By assumption, the transition matrix from basis 1 to basis 2 is upper triangular and Toeplitz. For notational convenience denote basis 1 by $\{u_i\}_{i=0}^d$ and basis 2 by $\{v_i\}_{i=0}^d$. By Proposition 12.8 there exists $A \in \text{End}(V)$ such that

$$\begin{aligned} Au_i &= u_{i-1} & (1 \leq i \leq d), & & Au_0 &= 0, \\ Av_i &= v_{i-1} & (1 \leq i \leq d), & & Av_0 &= 0. \end{aligned} \tag{219}$$

Consequently A lowers \mathcal{D}_1 and \mathcal{D}_2 . Similarly there exists $B \in \text{End}(V)$ that lowers \mathcal{D}_3 and \mathcal{D}_4 . Also there exists $C \in \text{End}(V)$ that lowers \mathcal{D}_5 and \mathcal{D}_6 . By our assumption concerning the three antidiagonal transition matrices, the decompositions $\mathcal{D}_2, \mathcal{D}_4, \mathcal{D}_6$ are the inversions of $\mathcal{D}_3, \mathcal{D}_5, \mathcal{D}_1$, respectively. By these comments A, B, C raise $\mathcal{D}_6, \mathcal{D}_2, \mathcal{D}_4$ respectively. Observe that \mathcal{D}_2 is lowered by A and raised by B ; therefore A, B form an LR pair on V . Similarly \mathcal{D}_4 is lowered by B and raised by C ; therefore B, C form an LR pair on V . Also \mathcal{D}_6 is lowered by C and raised by A ; therefore C, A form an LR pair on V . By these comments A, B, C form an LR triple on V . We now show that basis 2 is an (A, B) -basis of V . We just mentioned that \mathcal{D}_2 is lowered by A and raised by B . Therefore \mathcal{D}_2 is the (A, B) -decomposition of V . Now using (219) and Definition 3.21, we see that basis 2 is an (A, B) -basis of V . We have shown that basis 2 meets the requirements of the table in (ii). One similarly shows that bases 1, 3, 4, 5, 6 meet these requirements.

(ii) \Rightarrow (i) By assumption basis 1 is an (A, C) -basis of V , and basis 2 is an (A, B) -basis of V . By Lemma 13.41, the transition matrix from basis 1 to basis 2 is upper triangular

and Toeplitz. By assumption basis 3 is a (B, A) -basis of V . So the inversion of basis 3 is an inverted (B, A) -basis of V . By Lemma 3.48, the transition matrix from basis 2 to the inversion of basis 3 is diagonal. Recall that \mathbf{Z} is the transition matrix between basis 3 and its inversion. By these comments the transition matrix from basis 2 to basis 3 is antidiagonal. The remaining assertions of part (i) are similarly obtained.

Assume that (i), (ii) hold. From the construction of A in the proof of (i) \Rightarrow (ii) above, we find that A is uniquely determined by the sequence (218). Similarly B and C are uniquely determined by the the sequence (218). \square

Theorem 34.2. *Suppose we are given six invertible matrices in $\text{Mat}_{d+1}(\mathbb{F})$:*

$$\begin{matrix} D_1, & D_2, & D_3 & (\text{diagonal}), \end{matrix} \quad (220)$$

$$\begin{matrix} T_1, & T_2, & T_3 & (\text{upper triangular Toeplitz}). \end{matrix} \quad (221)$$

Then the following (i), (ii) are equivalent.

(i) $T_1 D_1 \mathbf{Z} T_2 D_2 \mathbf{Z} T_3 D_3 \mathbf{Z} \in \mathbb{F} I.$

(ii) *There exists an LR triple A, B, C over \mathbb{F} for which the matrices (220), (221) are transition matrices of the following kind:*

transition matrix	from a basis of type	to a basis of type
T_1	(A, C)	(A, B)
D_1	(A, B)	inv. (B, A)
T_2	(B, A)	(B, C)
D_2	(B, C)	inv. (C, B)
T_3	(C, B)	(C, A)
D_3	(C, A)	inv. (A, C)

Suppose (i), (ii) hold. Then the LR triple A, B, C is uniquely determined up to isomorphism by the sequence $D_1, D_2, D_3, T_1, T_2, T_3$.

Proof. (i) \Rightarrow (ii) We invoke Theorem 34.1. Multiplying D_1 by a nonzero scalar in \mathbb{F} if necessary, we may assume without loss that $T_1 D_1 \mathbf{Z} T_2 D_2 \mathbf{Z} T_3 D_3 \mathbf{Z} = I$. By linear algebra there exist six bases of V as in (218) such that the transition matrix

from basis	to basis	is
1	2	T_1
2	3	$D_1 \mathbf{Z}$
3	4	T_2
4	5	$D_2 \mathbf{Z}$
5	6	T_3
6	1	$D_3 \mathbf{Z}$

By construction, these six bases satisfy Theorem 34.1(i). Therefore they satisfy Theorem 34.1(ii). The LR triple A, B, C mentioned in Theorem 34.1(ii) satisfies condition (ii) of the present theorem.

(ii) \Rightarrow (i) Associated with the LR triple A, B, C are the upper triangular Toeplitz matrices T, T', T'' from Definition 13.44, and the diagonal matrices D, D', D'' from Definition 13.69. Using Lemma 3.22 and Definition 13.44(ii) we find $T_1 \in \mathbb{F}T'$. Similarly $T_2 \in \mathbb{F}T''$ and $T_3 \in \mathbb{F}T$. Using Lemma 3.48 we find $D_1 \in \mathbb{F}D$. Similarly $D_2 \in \mathbb{F}D'$ and $D_3 \in \mathbb{F}D''$. By these comments and Proposition 13.72 we obtain $T_1 D_1 \mathbf{Z} T_2 D_2 \mathbf{Z} T_3 D_3 \mathbf{Z} \in \mathbb{F}I$.

Assume that (i), (ii) hold. Consider the matrices D, D', D'' and T, T', T'' from the proof of (ii) \Rightarrow (i) above. By Proposition 15.9 the LR triple A, B, C is uniquely determined up to isomorphism by the sequence D, D', D'', T, T', T'' . The matrix D is obtained from D_1 by dividing D_1 by its $(0, 0)$ -entry. So D is determined by D_1 . The matrices D', D'', T, T', T'' are similarly determined by D_2, D_3, T_3, T_1, T_2 , respectively. By these comments the LR triple A, B, C is uniquely determined up to isomorphism by the sequence $D_1, D_2, D_3, T_1, T_2, T_3$. \square

Theorem 34.3. *Let $A, B, C \in \text{End}(V)$. Then A, B, C form an LR triple on V if and only if the following (i)–(iv) hold:*

- (i) *each of A, B, C is Nil;*
- (ii) *the flag $\{A^{d-i}V\}_{i=0}^d$ is raised by B, C ;*
- (iii) *the flag $\{B^{d-i}V\}_{i=0}^d$ is raised by C, A ;*
- (iv) *the flag $\{C^{d-i}V\}_{i=0}^d$ is raised by A, B .*

Proof. By Proposition 3.46 and Definition 13.1. \square

35 Appendix I: The nonbipartite LR triples in matrix form

In this section we display the nonbipartite equitable LR triples in matrix form.

Let V denote a vector space over \mathbb{F} with dimension $d + 1$. Let A, B, C denote a nonbipartite equitable LR triple on V , with parameter array (44), trace data (50), and Toeplitz data (54). By Definition 17.1 we have $\alpha_i = \alpha'_i = \alpha''_i$ for $0 \leq i \leq d$, and by Lemma 17.3 we have $\beta_i = \beta'_i = \beta''_i$ for $0 \leq i \leq d$. By (57) we have $\beta_1 = -\alpha_1$. By Lemma 17.6 we have $\varphi_i = \varphi'_i = \varphi''_i$ for $1 \leq i \leq d$, and $a_i = a'_i = a''_i = \alpha_1(\varphi_{d-i+1} - \varphi_{d-i})$ for $0 \leq i \leq d$. Recall from Definition 18.2 that A, B, C is normalized if and only if $\alpha_1 = 1$. By Proposition 13.39 we have the following.

With respect to an (A, B) -basis for V the matrices representing A, B, C are

$$A : \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & 0 & 1 & \\ & & 0 & \cdot \\ & & & \cdot \\ & & & \cdot & 1 \\ \mathbf{0} & & & & 0 \end{pmatrix}, \quad B : \begin{pmatrix} 0 & & & \mathbf{0} \\ \varphi_1 & 0 & & \\ & \varphi_2 & 0 & \\ & & \cdot & \cdot \\ & & & \cdot & \cdot \\ \mathbf{0} & & & \varphi_d & 0 \end{pmatrix},$$

$$C : \begin{pmatrix} a_0 & \varphi_d/\varphi_1 & & & \mathbf{0} \\ \varphi_d & a_1 & \varphi_{d-1}/\varphi_2 & & \\ & \varphi_{d-1} & a_2 & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \varphi_1/\varphi_d \\ \mathbf{0} & & & \varphi_1 & a_d \end{pmatrix}.$$

With respect to an inverted (A, B) basis for V the matrices representing A, B, C are

$$A : \begin{pmatrix} 0 & & & \mathbf{0} \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \cdot & \cdot \\ & & & \cdot \\ \mathbf{0} & & & 1 & 0 \end{pmatrix}, \quad B : \begin{pmatrix} 0 & \varphi_d & & \mathbf{0} \\ & 0 & \varphi_{d-1} & \\ & & 0 & \cdot \\ & & & \cdot & \cdot \\ & & & & \cdot & \varphi_1 \\ \mathbf{0} & & & & & 0 \end{pmatrix},$$

$$C : \begin{pmatrix} a_d & \varphi_1 & & & \mathbf{0} \\ \varphi_1/\varphi_d & a_{d-1} & \varphi_2 & & \\ & \varphi_2/\varphi_{d-1} & a_{d-2} & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \varphi_d \\ \mathbf{0} & & & \varphi_d/\varphi_1 & a_0 \end{pmatrix}.$$

With respect to a (B, A) basis for V the matrices representing A, B, C are

$$A : \begin{pmatrix} 0 & & & \mathbf{0} \\ \varphi_d & 0 & & \\ & \varphi_{d-1} & 0 & \\ & & \cdot & \cdot \\ & & & \cdot \\ \mathbf{0} & & & \varphi_1 & 0 \end{pmatrix}, \quad B : \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & 0 & 1 & \\ & & 0 & \cdot \\ & & & \cdot & \cdot \\ & & & & \cdot & 1 \\ \mathbf{0} & & & & & 0 \end{pmatrix},$$

$$C : \begin{pmatrix} a_d & \varphi_1/\varphi_d & & & \mathbf{0} \\ \varphi_1 & a_{d-1} & \varphi_2/\varphi_{d-1} & & \\ & \varphi_2 & a_{d-2} & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \varphi_d/\varphi_1 \\ \mathbf{0} & & & \varphi_d & a_0 \end{pmatrix}.$$

With respect to an inverted (B, A) basis for V the matrices representing A, B, C are

$$A : \begin{pmatrix} 0 & \varphi_1 & & & \mathbf{0} \\ & 0 & \varphi_2 & & \\ & & 0 & \cdot & \\ & & & \cdot & \cdot \\ & & & & \varphi_d \\ \mathbf{0} & & & & 0 \end{pmatrix}, \quad B : \begin{pmatrix} 0 & & & & \mathbf{0} \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ \mathbf{0} & & & & 1 & 0 \end{pmatrix},$$

$$C : \begin{pmatrix} a_0 & \varphi_d & & & \mathbf{0} \\ \varphi_d/\varphi_1 & a_1 & \varphi_{d-1} & & \\ & \varphi_{d-1}/\varphi_2 & a_2 & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ \mathbf{0} & & & & \varphi_1/\varphi_d & a_d \end{pmatrix}.$$

36 Appendix II: The bipartite LR triples in matrix form

In this section we display the bipartite LR triples in matrix form.

Let V denote a vector space over \mathbb{F} with dimension $d+1$. Let A, B, C denote a bipartite LR triple on V , with parameter array (44). By Proposition 13.39 we have the following.

With respect to an (A, B) -basis for V the matrices representing A, B, C are

$$A : \begin{pmatrix} 0 & 1 & & & \mathbf{0} \\ & 0 & 1 & & \\ & & 0 & \cdot & \\ & & & \cdot & \cdot \\ & & & & 1 \\ \mathbf{0} & & & & 0 \end{pmatrix}, \quad B : \begin{pmatrix} 0 & & & & \mathbf{0} \\ \varphi_1 & 0 & & & \\ & \varphi_2 & 0 & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ \mathbf{0} & & & & \varphi_d & 0 \end{pmatrix},$$

$$C : \begin{pmatrix} 0 & \varphi'_d/\varphi_1 & & & \mathbf{0} \\ \varphi''_d & 0 & \varphi'_{d-1}/\varphi_2 & & \\ & \varphi''_{d-1} & 0 & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ \mathbf{0} & & & & \varphi'_1/\varphi_d & \varphi''_1 & 0 \end{pmatrix}.$$

With respect to an inverted (A, B) basis for V the matrices representing A, B, C are

$$A : \begin{pmatrix} 0 & & & \mathbf{0} \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \\ \mathbf{0} & & & 1 & 0 \end{pmatrix}, \quad B : \begin{pmatrix} 0 & \varphi_d & & \mathbf{0} \\ & 0 & \varphi_{d-1} & \\ & & 0 & \ddots \\ & & & \ddots & \varphi_1 \\ \mathbf{0} & & & & 0 \end{pmatrix},$$

$$C : \begin{pmatrix} 0 & \varphi_1'' & & \mathbf{0} \\ \varphi_1'/\varphi_d & 0 & \varphi_2'' & \\ & \varphi_2'/\varphi_{d-1} & 0 & \ddots \\ & & \ddots & \ddots & \ddots \\ \mathbf{0} & & & \varphi_d'/\varphi_1 & 0 \end{pmatrix}.$$

With respect to a (B, A) basis for V the matrices representing A, B, C are

$$A : \begin{pmatrix} 0 & & & \mathbf{0} \\ \varphi_d & 0 & & \\ & \varphi_{d-1} & 0 & \\ & & \ddots & \ddots \\ \mathbf{0} & & & \varphi_1 & 0 \end{pmatrix}, \quad B : \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & 0 & 1 & \\ & & 0 & \ddots \\ & & & \ddots & 1 \\ \mathbf{0} & & & & 0 \end{pmatrix},$$

$$C : \begin{pmatrix} 0 & \varphi_1''/\varphi_d & & \mathbf{0} \\ \varphi_1' & 0 & \varphi_2''/\varphi_{d-1} & \\ & \varphi_2' & 0 & \ddots \\ & & \ddots & \ddots & \ddots \\ \mathbf{0} & & & \varphi_d''/\varphi_1 & 0 \end{pmatrix}.$$

With respect to an inverted (B, A) basis for V the matrices representing A, B, C are

$$A : \begin{pmatrix} 0 & \varphi_1 & & \mathbf{0} \\ & 0 & \varphi_2 & \\ & & 0 & \ddots \\ & & & \ddots & \varphi_d \\ \mathbf{0} & & & & 0 \end{pmatrix}, \quad B : \begin{pmatrix} 0 & & & \mathbf{0} \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \\ \mathbf{0} & & & 1 & 0 \end{pmatrix},$$

$$C : \begin{pmatrix} 0 & \varphi_d' & & \mathbf{0} \\ \varphi_d''/\varphi_1 & 0 & \varphi_{d-1}' & \\ & \varphi_{d-1}''/\varphi_2 & 0 & \ddots \\ & & \ddots & \ddots & \ddots \\ \mathbf{0} & & & \varphi_1''/\varphi_d & 0 \end{pmatrix}.$$

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